ii) Show that the vector $\vec{F} = (4xy - z^3)i + 2x^2 j - 3xz^2 k$ is irrotational. Show that $\vec{F}$ can be expressed as the gradient of a scalar function ϕ. 1+4

c) i) If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \cdots + \frac{c_n}{n+1} = 0$, where $c_i \in \mathbb{R}$ for $i = 0,1,..,n$, show that the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ has atleast one real root between 0 and 1. 4

ii) A function $f: \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function on $\mathbb{R}$. 3

iii) Prove that $div\left[(\vec{r} \times \vec{a}) \times \vec{b}\right] = -2\vec{b}.\vec{a}$. 3

d) i) Obtain the Maclaurin's infinite series expansion of $(1+x)^m$, where m is any real number other than positive integer and $|x| < 1$.

ii) If $f: [0,1] \to \mathbb{R}$ is such that $f''(x) < 0$ in $[0,1]$ and if $\phi(x) = f(x) + f(1-x)$ in $[0,1]$, show that ϕ is monotone increasing in $\left[0, \frac{1}{2}\right]$ and it is monotone decreasing in $\left[\frac{1}{2}, 1\right]$. 5+5

U.G. 3rd Semester Examination - 2024

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-05

(Theory of Real & Vector Functions)

[CBCS]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

i) Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{R}$ and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. If $f(0) = 0$, then show that $f(x) > 0$ for all $x \in \mathbb{R}$.

ii) A function $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x) = 0, \forall x \in \mathbb{Q}$. Prove that $f(x) = 0, \forall x \in \mathbb{R}$.

iii) Give an example of a function f which satisfies the intermediate-value property on a closed and bounded interval [a, b], but is not continuous on [a, b].

iv) Let a function $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x^2, & \text{if } x\in\mathbb{Q} \\ 0, & \text{if } x\in\mathbb{R}/\mathbb{Q} \end{cases}.$$

Then show that $f'(0)=0$.

v) Find grad $\log|\vec{r}|$.

vi) Give an example of a function which satisfies all the conditions of Rolle's theorem, where the derivative of the function vanishes at two distinct interior points.

vii) Determine the constants a, b, c so that the vector

$$\bar{A}=(x+2y+az)i+(bx-3y-z)j+(4x+cy+2z)k$$

is irrotational.

viii) State Rolle's theorem.

ix) Find $\lim_{x\to 0} e^x\, sgn(x+[x])$, where the signum function is defined as

$$sgn(x)=\begin{cases} \frac{|x|}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}.$$

x) Give an example of a function f which is continuous on a closed interval I, but f(I) is not a closed interval.

xi) If $f(x)=\max\{x,x^{-1}\}$,, for x > 0, then obtain the value of $f(c).f(c^{-1})$ at $c>0$.

xii) A particle moves along the curve $x=2t^2, y=t^2-4t, z=3t-5$. Determine the velocity and acceleration at any time t = 1.

xiii) If $f(x)=-\sqrt{1-x^2}$, then show that $f'(x)$ assumes every real value as x runs over (–1,1).

xiv) If the vectors $\vec{A}$ and $\vec{B}$ are irrotational, then show that the vector $\vec{A}\times\vec{B}$ is solenoidal.

xv) If f is derivable on [0,1], then show by Cauchy's mean value theorem that $f(1)-f(0)=\frac{f'(x)}{2x}$ has at least one solution in (0,1).

2. Answer any **four** questions: 5×4=20

a) i) The functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ both are continuous on $\mathbb{R}$. Then show that the set $S=\{x\in\mathbb{R}: f(x)=g(x)\}$ is a closed set in $\mathbb{R}$.

ii) Construct a function $f:[0,\frac{\pi}{2}]\to\mathbb{R}$ which is unbounded on the closed interval and discontinuous only at the point $\frac{\pi}{2}$. 4+1

b) Let $D\subset\mathbb{R}$. A function $f: D\to\mathbb{R}$ is uniformly continuous on D if and only if for every pair of sequences $\{x_n\}$ and $\{y_n\}$ in D satisfying $\lim_{n\to\infty}|f(x_n)-f(y_n)|=0$ holds. 5

c) i) Let $f:[a,b]\to\mathbb{R}$ be a thrice differentiable function on $[a, b]$. If $f(a)=f(b)=f'(a)=f'(b)=0$, then prove that $f'''(c)=0$ for some $c\in(a,b)$.

ii) Show that between any two roots of $e^x \sin x = 1$, , there is a real root of $e^x \cos x + 1 = 0$. 3+2

d) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(x+y)=f(x).f(y)\ \forall x,y\in\mathbb{R}$. If $f(1)=c$ then show that $f(x)=cx, \forall x\in\mathbb{R}$. 5

e) i) If $f(x)=\begin{cases} 2x+3, if\ x\le 1 \\ ax^2+bx, if\ x>1 \end{cases}$,

where a and b are real constants and if f is derivable everywhere, then find the values of a and b.

ii) Show that $f(x)=[x]$ in $[0,2]$ is not derivable of any function. 3+2

f) If f is derivable in $[a,b]$ and $f'(a)$ and $f'(b)$ are of opposite signs, then show that there exists at least one point $c\in(a,b)$ such that $f'(c)=0$. 5

3. Answer any **two** questions: 10×2=20

a) i) If $f(x)=\sin x$, then prove that $\lim_{h\to 0^+}\theta=\frac{1}{\sqrt{3}}$, where θ is given by

$$f(h)=f(0)+hf'(\theta h), 0<\theta<1.$$

ii) If $\bar{w}$ is a constant vector, $\bar{r}$ and $\bar{s}$ are vectors functions of a scalar variable t and if $\frac{d\bar{r}}{dt}=\bar{w}\times\bar{r}$, $\frac{d\bar{s}}{dt}=\bar{w}\times\bar{s}$, then show that $\frac{d}{dt}(\bar{r}\times\bar{s})=\bar{w}\times(\bar{r}\times\bar{s})$.

iii) Use Cauchy's mean value theorem to evaluate

$$\lim_{h\to 1}\frac{\cos\frac{1}{2}\pi x}{\log\left(\frac{1}{x}\right)}.$$ 4+3+3

b) i) A function $f:[0,1]\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases} x, & x\ is\ rational\ in\ [0,1] \\ 1-x, & x\ is\ irrational\ in\ [0,1]. \end{cases}$$

Show that (i) f is injective on [0,1], (ii) f assumes every real number in [0,1], (iii) f is continuous at $\frac{1}{2}$ and discontinuous at every other point in [0,1]. 2+1+2