U.G. 1st Semester Examination - 2023

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-02

(Algebra)

[Old CBCS Syllabus]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

GROUP–A

1. Answer any **ten** questions: 2×10=20

   a) In any group G if $a^3 = e$ and $aba^{-1} = b^2$ for $a, b \in G$, then find the order of b.

   b) Expand $\cos^7\theta$ in a series of cosines of muiltiples of θ.

   c) Find all real values of λ for which the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$ is 2.

[Turn over]

d) Prove that $7^{2n} - 48n - 1$ is divisible by 2304 for every natural number n.

e) What is the remainder when $1! + 2! + 3! + \dots + 100!$ is divided by 15?

f) Discuss the nature of roots of the equation $x^4 + 4x^3 - 12x^2 - 32x + k$ for all values of k.

g) If $a \mid bc$ and $\gcd(a, b) = 1$, then prove that $a \mid c$.

h) Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}.$$

i) Prove that every prime greater than 3 can be written in the form $6n+1$ or $6n+5$.

j) Find the integers s, t such that

$$\gcd(33, 57) = 33s + 57t.$$

k) If α, β, γ are roots of the equation $x^3 + qx + r = 0$ then find the equation whose roots are $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$.

l) Find λ such that $A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ is an orthogonal matrix.

m) If z is a complex number such that $|z|=2$ then show that the point $z+\frac{1}{z}$ lies on an ellipse of eccentricity $\frac{4}{5}$ in the complex plane.

n) In a group (G, o), show that $a \circ b = a \circ c \Rightarrow b = c$ for all $a, b, c \in G$.

o) Find the sum of all 1729th roots of unity.

GROUP–B

2. Answer any **four** questions: $5\times4=20$

a) Reduce the matrix $A=\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row reduced echelon form and hence find its rank. 4+1

b) Define equivalence relation on a set S. Show that the relation

$$\rho=\{(a, b)\in\mathbb{Z}\times\mathbb{Z} : a-b \text{ is divisible by } 5\}$$

is an equivalence relation on the set $\mathbb{Z}$ of all integers. 2+3

c) i) Prove that $2^n 3^{2n} - 1$ is always divisible by 17.

ii) Prove that for every integer n, $n^3 \pmod 6 = n \pmod 6$. 2+3

d) i) If $n = p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes, prove that

$$\sum_{d|n} \left|\mu(d)\right| = 2^k .$$

ii) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$, prove that

$$\sum_{d|n} \mu(d)\phi(d) = (2-p_1)(2-p_2)\ldots(2-p_k).$$

2+3

e) Define a group. Give an example of finite and infinite group. Does the set $\mathbb{Z}$ of all integers form a group under the operation defined as $a \circ b = a - b$ for all $a, b \in \mathbb{Z}$? Justify your answer. 2+1+2

f) i) If $V_n = a^n + b^n$, where a and b are the roots of $x^2 + x + 1$, what is the value of

$$\sum_{n=0}^{1729} (-1)^n . V_n ?$$

ii) Prove that for any natural number n,

$$2\left(\sqrt{n+1}-1\right) < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

2+3

GROUP–C

3. Answer any **two** questions: 10×2=20

a) i) Prove or give a counter example to the following statement:

If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

ii) If $(A \mid b)$ is in reduced row echelon form, prove that A is also in reduced row echelon form.

iii) Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix},$$

find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

3+3+4

b) i) Let $f: A \to B$ and $g: B \to C$ be two mappings such that $g \circ f$ is injective and f is surjective. Prove that g is injective.

ii) In a group $(G, \circ)$, prove that $(a^{-1})^{-1} = a$ and $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ for all $a, b \in G$.

iii) Show that the residue classes [1], [3], [5], [7] modulo 8 form a multiplicative group.

3+3+4

c) i) Investigate for what values of λ and μ the following equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have no solution, a unique solution and an infinite number of solutions.

ii) Obtain the normal form under congruence and find the rank, signature of the symmetric matrix $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{pmatrix}$.

5+5

d) i) For any set A, finite or infinite, let B^A be the set of all functions from A into the set $B = \{0, 1\}$. Show that the cardinality of B^A is the same as the cardinality of the set 2^A.

ii) If p is a prime, prove that

$$2p(p-3)!+1 \equiv 0 (\bmod p).$$

iii) Prove that 3, 5 and 7 are the only three consecutive odd integers which are prime.

3+3+4
