188/Math. UG/1st Sem/MATH-H-CC-T-02/22

U.G. 1st Semester Examination - 2022

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-02

(Algebra)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

GROUP–A

[Marks : 20]

1. Answer any **ten** questions: 2×10=20

a) Find the principal argument of $z = 1 + i\tan\frac{3\pi}{4}$.

b) Find the cube roots of -1.

c) Find the general value of i^i.

d) Show that $\sin\left\{i\log\left(\frac{a-ib}{a+ib}\right)\right\} = \frac{2ab}{a^2+b^2}$.

e) Find the equation whose roots are the cube of the roots of the equation $x^3 + 3x^2 + 2 = 0$.

[Turn Over]

f) If $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+3$, $x\in\mathbb{R}$, then find $f^{-1}(3)$ and $f^{-1}(7)$.

g) Give an example of a function which is neither one-one nor onto.

h) Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}.$$

i) If $a \mid bc$ and $\gcd(a, b)=1$, then prove that $a \mid c$.

j) Find the integers s, t such that

$$\gcd(33, 57)=33s+57t.$$

k) What is the remainder when $1!+2!+3!+\ldots+100!$ is divided by 15?

l) Find λ such that $A=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ is an orthogonal matrix.

m) Find all real values of λ for which the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}$ is 2.

n) In a group $(G, \circ)$, show that $a \circ b = a \circ c \Rightarrow b = c$ for all $a, b, c \in G$.

o) In any group G if $a^3 = e$ and $aba^{-1} = b^2$ for $a, b \in G$, then find the order of b.

GROUP–B

[Marks : 20]

2. Answer any **four** questions: 5×4=20

a) State De Moivre's theorem. If

$$z_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r} \left(r = 1, 2, 3, \ldots\right),$$

then prove that $z_1.z_2.z_3\ldots\infty = -1$. 2+3

b) Show that $n^2 < 2^n \ \forall \ n \geq 5$. 5

c) Solve by Cardan's method $x^3 - 12x + 65 = 0$. 5

d) Define equivalence relation on a set S. Show that the relation

$$\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b \text{ is divisible by } 5\}$$

is an equivalence relation on the set $\mathbb{Z}$ of all integers. 2+3

e) Define a group. Give an example of finite and infinite group. Does the set $\mathbb{Z}$ of all integers form a group under the operation defined as $a \circ b = a - b$ for all $a, b \in \mathbb{Z}$? Justify your answer. 2+1+2

f) Reduce the matrix $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ to a row reduced echelon form and hence find its rank. 4+1

GROUP–C

[Marks : 20]

3. Answer any **two** questions: 10×2=20

a) i) Show that the sum of 99*th* powers of the roots of the equation $x^5 - 1 = 0$ is zero. 5

ii) Apply Descarte's rule of signs to show that the equation $x^7 + 5x^4 - 3x + 1 = 0$ has at least four imaginary roots. 3

iii) If one of the roots of the equation $x^3 + px^2 + qx + r = 0$ equals the sum of the other two, then prove that $p^3 + 8r = 4pq$. 2

b) i) Let $f: A \to B$ and $g: B \to C$ be two mappings such that $g \circ f$ is injective and f is surjective. Prove that g is injective. 3

ii) In a group $(G, \circ)$, prove that $\left(a^{-1}\right)^{-1} = a$ and $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ for all $a, b \in G$. 3

iii) Show that the residue classes [1], [3], [5], [7] modulo 8 form a multiplicative group. 4

c) i) Investigate for what values of λ and μ the following equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

have no solution, a unique solution and an infinite number of solutions. 5

ii) Obtain the normal form under congruence and find the rank, signature of the symmetric matrix $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. 5

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