U.G. 1st Semester Examination - 2020

MATHEMATICS

[PROGRAMME]

Course Code : MATH-G-CC-T-01

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Give an example of non-removable discontinuity.

b) Show that $\lim_{x\to 0}\frac{1}{2+e^{\frac{1}{x}}}$ does not exists.

c) If $g(t)=t^2+\sqrt{t}$ and $f(x, y)=5x+3y-15$, find the domain where $g(f(x, y))$ is continuous.

d) Show that the function

$$f(x, y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & when\ \ x\neq y,\\ 0, & when\ \ x=y;\end{cases}$$

is not continuous at (0, 0).

e) Find the least distance of the point (0, 3) from the parabola $x^2=2y$.

f) Show that the following pair of curves $r^2\theta=a^2$ and $r=e^{\theta^2}$ cut orthogonally.

g) Examine whether that Rolle's theorem is applicable or not on the function $f(x)=x(x+3)e^{-\frac{1}{2}x}$ in [–3, 0].

h) If $f(x)=\begin{vmatrix}\cos x & \sin x & \cos x\\ \cos 2x & \sin 2x & 2\cos 2x\\ \cos 3x & \sin 3x & 3\cos 3x\end{vmatrix}$,

find $f'\left(\frac{\pi}{2}\right)$.

i) Find $\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

j) Show that $x=0$ is a cusp of the curve $y^3+3ax^2+x^3=0$.

k) Find the radius of curvature of the curve $r = ae^{\theta \cot\alpha}$ at θ.

l) Show that $x > \log(1+x) > x - \frac{1}{2}x^2$; $(x > 0)$.

m) Find the asymptotes of the curve $y^2(x-1) = x^3$.

n) If the area of a circle increases at a uniform rate, prove that the rate of increase of the perimeter varies as the radius.

o) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

2. Answer any **four** questions: 5×4=20

a) If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardiod $r = a(1+\cos\theta)$, which passes through the pole, then prove that $\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2$. 5

b) Show that the tangents drawn at the extremities of any chord of the cardiod $r = a(1+\cos\theta)$ which passes through the pole are perpendicular to each other. 5

c) If a straight line is drawn through the point $(a, 0)$ parallel to the asymptote of the cubic $(x-a)^3 - x^2y = 0$, prove that the portion of the line intercepted by the axes is bisected by the curve. 5

d) Find the 2nd degree Taylor polynomial for $\cos x$ around $x = \pi$. 5

e) Show that $\lim_{x\to 0} e^{\frac{-1}{x^2+y^2}} = 1$. 5

f) If $y = \sin\left(m\cos^{-1}\sqrt{x}\right)$, then prove that $\lim_{x\to 0}\frac{y_{n+1}}{y_n} = \frac{4n^2 - m^2}{4n+2}$. 5

g) Find the expansion of $(1+x)^n$ in a power series of x and indicate the range of validity of the expansion. 5

3. Answer any **two** questions: 10×2=20

a) i) If $y = \left(\sinh^{-1}x\right)^2$, prove that $\left(x^2+1\right)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$. 5

ii) If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$. 5

b) i) If $f(x+y) = f(x) + f(y)$ for all x and y and $f(x)$ is continuous at $x = 0$. Then show that $f(x)$ is continuous for all values of x. 5

ii) If H be a homogeneous function in x, y, z of degree n then show that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)^5 + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right) = 0,$$

if $u = \left(x^2 + y^2 + z^2\right)^{-\frac{n+1}{2}}$. 5

c) i) State and prove the Taylor's theorem with Cauchy's form of remainder. 5

ii) Trace the following curve:

$$y^2(x+3a) = x(x-a)(x-2a). \quad 5$$

d) i) Determine a such that $\lim\limits_{x \to 0} \frac{e^x - ae^{x\cos x}}{x - \sin x}$ is finite. What is the value of this limit? 5

ii) Find the greatest value of $x^m y^n$ $(x, y > 0)$ where $x + y = k$, k is a constant. 5
