U.G. 1st Semester Examination - 2020

MATHEMATICS

[HONOURS]

Generic Elective Course (GE)

Course Code : MATH-H-GE-T-01

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Write the geometrical interpretation of Lagrange's M.V.T.

b) Find the asymptotes of the curve $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$.

c) What do you mean by jump discontinuity of a function? Give an example of it.

d) Examine whether the L'Hospital rule is applicable on $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. If yes, then find the value of it.

[Turn over]

e) If $u = \tan^{-1}\frac{x^3+y^3}{x^2+y^2}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u$.

f) Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'\left(\theta - \frac{\pi}{2}\right)$.

g) Give an example of a function *f* which satisfies the intermediate-value property on a closed and bounded interval [*a*, *b*] but is not continuous on [*a*, *b*].

h) Prove that $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$.

i) Find the derivative of the function $f(x) = x^{\alpha}$, $x > 0$ and $\alpha \in \mathbb{R}$.

j) Show that for no value of *k*, $f(x) = x^3 - 3x + k$ has two distinct zeros in (0, 1).

k) Show that if *c* is an interior point of the domain of a function $f(x)$ and $f'(c) = 0$, then the function has a maxima or a minima at *c* according as $f''(c)$ is negative or positive.

l) Show that the curve $2y^2 = x^2 y + x^3$ has a cusp of the first species at the origin.

m) Show that $\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$ does not exist.

2. Answer any **four** questions: 5×4=20

a) If $y=\frac{1}{x^2+a^2}$, then prove that $y_n=(-1)^n\frac{n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$, where $\cot\theta=\frac{x}{a}$.

b) Find the maximum and minimum values of the function $\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$, $0\le x\le\pi$.

c) Prove that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches to $\frac{1}{n+1}$ as h approaches zero, provided that $f^{n+1}(x)$ is continuous and different from zero at $x=a$.

d) Show that the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with respect to a focus is $\frac{b^2}{p^2}=\frac{2a}{r}-1$.

e) Find the curvature at the origin of each of the two branches of the curve

$$y(ax+by)=cx^3+ex^2y+fxy^2+gy^3.$$

f) Let $f(x, y)=\frac{2xy}{\sqrt{x^2+y^2}}$, $(x, y)\neq(0, 0)$ and $f(0,0)=0$. Then show that $f_{xy}(0, 0)=f_{yx}(0, 0)$.

3. Answer any **two** questions: 10×2=20

a) i) Obtain Maclaurin's series expansion of $\log_e(1+x)$ over the open interval [–1, 1]. 6

ii) If $f'(x)=(x-a)^{2n}(x-b)^{2m+1}$ where m, n are positive integers, show that f has neither a maximum nor a minimum at about f has a minimum at b. 4

b) i) Prove that the function f defined by $f(x)=\sin\frac{1}{x}$ is continuous for all $x>0$ but is not uniformly continuous there. 5

ii) If a real valued function $f(x)$ is continuous in a closed interval [a, b] then show that $f(x)$ is uniformly continuous in [a, b] also. 5

c) i) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that

$$\left(x^2-1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0.$$

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ii) A function f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$ and $f(c) < 0$ for some c in $[a, b]$. Prove that there is at least one point α in (a, b) for which $f''(\alpha) > 0$. 4

d) i) If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, where u is a function of x, y, z, then prove that

$$(u_x)^2 + (u_y)^2 + (u_z)^2 = 2\left(xu_x + yu_y + xu_z\right).$$

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ii) Show that the semi-vertical angle of a right circular cone of maximum possible volume and of the given curved surface is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$. 5
