U.G. 1st Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MATH(H)/CC-2-T

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

i) Find the value of $\left(1+i\sqrt{3}\right)^{30}$.

ii) If $x+\frac{1}{x}=2\cos\frac{\pi}{7}$, then find the value of $x^7+\frac{1}{x^7}$.

iii) Find the remainder when $3x^4-4x^3+2x^2-9x+1$ is divided by $2x+1$.

iv) Find the value of $\left(4+3\sqrt{-20}\right)^{\frac{1}{2}}+\left(4-3\sqrt{-20}\right)^{\frac{1}{2}}$.

v) If α, β and γ are the roots of $x^3+px+q=0$, then find $\sum\frac{1}{\alpha^2-\beta\gamma}$.

vi) If the roots of the equation $x^3-3x^2+ax+b=0$ are in A.P., then find the value of a+b.

vii) If f, g: $\mathbb{R}\to\mathbb{R}$ are two functions such that $f(x)=2x+1$ and $g(x)=x^2-2$, then find $g\circ f$.

viii) Find the inverse of the matrix:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

ix) Find the equation whose roots exceed by 2 from the roots of the equation

$$4x^4+32x^3+83x^2+76x+21=0.$$

x) If a, b, c are all positive real numbers such that a+b+c=1, then prove that

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{9}{2}.$$

xi) For what value of k, the vectors $(k, 1, 0)$, $(1, k, 1)$ and $(0, 1, k)$ in $\mathbb{R}^3$ are linearly dependent?

xii) Show that 5^n-1 is divisible by 4 for all positive integer n.

xiii) Give an example of a mapping which is

a) One-to-one but not onto

b) Onto but not one-to-one.

xiv) State Descarte's rule of signs.

xv) Is the matrix multiplication commutative, in general? Justify.

2. Answer any **four** questions: 5×4=20

a) Determine the rank of the following matrix for different values of λ:

$$\begin{pmatrix} \lambda & 1 & 0 \\ 3 & \lambda-2 & 1 \\ 3(\lambda+1) & 0 & \lambda+1 \end{pmatrix}$$

b) If $x^4-14x^2+24x-k=0$ has four real and unequal roots, prove that k must lie between 8 and 11.

c) For a suitable h apply the transformation x=y+h to remove the term containing x^2 from the equation

$$x^3-15x^2-33x+847=0$$

and then solve the transformed equation by Cardan's method. Hence find the roots of the equation.

d) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

and hence find A^{-1}.

e) Prove that the necessary and sufficient condition for a mapping to be invertible is that it is one-to-one and onto.

f) If a and b are two integers with b>0, then prove that there exists unique integers q and r such the a=bq+r with $0 \le r < b$.

g) Discuss consistency and solutions of the following system of linear equations:

$$x + ay + az = 1$$

$$ax + y + 2az = -4$$

$$ax - ay + 4z = 2$$

for different values of a.

Answer any **two** questions from **Q. No. 3** to **Q. No. 6:**

10×2=20

3. a) If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$ and $\cos\phi = \frac{1}{2}\left(b + \frac{1}{b}\right)$, then show that $\cos(\theta+\phi)$ is one of the values of $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$.

b) If a, b, c, d are positive reals such that a+b+c+d=1 then prove that

$$\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \ge \frac{4}{7}.$$

c) Expand $\cos^7\theta$ in a series of cosines of multiple of θ.

d) Show that

$$W = \{(x, y, z) : x \ge 0,\ x, y, z \in \mathbb{R}\}$$

is not a subspace of $\mathbb{R}^3$.

3+3+2+2

4. a) Let R be an equivalence relation on a set S and for $a \in S$, let [a] denote the R-equivalence class of a in S. For any two elements $x, y \in S$ if $[x] \ne [y]$, then show that $[x] \cap [y] = \phi$.

b) Prove that the product of any m consecutive integers is divisible by m.

c) Solve the biquadratic equation:

$$x^4-6x^2+16x-15=0.$$ 3+3+4

5. a) State and prove Cayley-Hamilton Theorem.

b) Reduce the matrix A to row reduced echelon form and then find its rank

$$A=\begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}.$$

c) Prove that $1!\,3!\,5!\ldots(2n-1)!>(n!)^n$.

d) If a, b, c are positive reals and $abc=k^3$, then prove that $(1+a)(1+b)(1+c)\geq(1+k)^3$.

3+3+2+2

6. a) If z_1 and z_2 be two complex numbers, then show that

i) $|z_1 \pm z_2| \geq ||z_1|-|z_2||$

ii) $z_1\bar{z}_2+\bar{z}_1z_2 \leq 2|z_1||z_2|$

where $\bar{z}$ is the conjugate of z.

b) For any two non-empty sets X and Y, let $f:X\to Y$ be a mapping such that

$$f(A\cap B)=f(A)\cap f(B)$$

for all non-empty sets A and B of X. Prove that f is injective.

c) Show that the open intervals (0, 1) and $(0, \infty)$ have the same cardinality. (2+2)+3+3
