iii) If $\varphi$ and $\psi$ are both continuous in [a, b] and are both derivable in (a, b) and if $\varphi'$ and $\psi'$ never vanish, then prove that

$$\frac{\varphi(\xi)-\varphi'(a)}{\psi(b)-\psi'(\xi)}=\frac{\varphi'(a)}{\psi'(\xi)},\ a<\xi<b.$$

3+3+4

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## U.G. 2nd Semester Examination - 2024

# MATHEMATICS

**[PROGRAMME]**

**Course Code : MATH-G-CC-T-02**

**(Calculus & Differential Equations)**

**[CBCS]**

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.*

*The symbols and notations have their usual meanings.*

1. Answer any **ten** questions: 2×10=20

a) If $y=\sqrt{sinx+\sqrt{sinx+\sqrt{sinx\ .....\infty}}}$ Show that $\frac{dy}{dx}=\frac{cosx}{2y-1}$.

b) Evaluate the following limit (if exist):

$$\lim_{x\to 0}\frac{3x+|x|}{7x-5|x|}.$$

c) Obtain the differential equation corresponding, to the primitive $(x-\alpha)^2+(x-\beta)^2=r^2$ where r is fixed constant and $\alpha,\ \beta$ are arbitrary constants.

d) Prove that the curve $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut orthogonally.

e) If $f(x) = \cos x$, then prove that $\lim_{h\to 0^+} \theta = \frac{1}{2}$ where $\theta$ is given by $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$

f) If $y = \left(x + \sqrt{1+x^2}\right)^m$, find the value of $y_n(0)$.

g) If $S_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} dx$, n being an integer, show that $S_{n+1} = S_n = \frac{\pi}{2}$.

h) Let $a \in \mathbb{R}$ and a real function f be such that $f''(x)$ exists in [a – h, a + h] for some h> 0. Prove that $\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(c)$ for some $c \in [a-h, a+h]$.

i) Prove that $\left(\frac{1}{3x^3y^3}\right)$ is an integrating factor of $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$.

j) If $y = \frac{1}{x^2 - a^2}$, then find $y_n$.

k) A function $f: [0, 1] \to R$ is defined by

$$f(x) = x, x \text{ is rational in } [0,1]$$
$$= 1 - x, x \text{ is irrational in } [0,1].$$

Show that f is continuous at $\frac{1}{2}$ and discontinuous at every other point in [0, 1].

l) Show that $x > \sin x$ for all $x$ in $0 < x < \frac{\pi}{2}$.

m) If $y = x^{n-1} \log x$ then prove that $y_n = \frac{(n-1)!}{x}$.

n) If $I_n = \int e^{-x}x^n dx$, show that $I_n = -e^{-x}x^n + nI_{n-1}$.

o) Apply Euler's Theorem to show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6f$ where $f(x, y) = x^4y^2 \sin^{-1}\frac{y}{x}$.

2. Answer any **four** questions: 5×4=20

a) Let $f: [a, b] \to \mathbf{R}$ be a function such that $f^{(n-1)}$ is continuous in [a, b ] and $f^{(n)}$ exists in (a, b). Show that there exists a $\theta \in (0, 1)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) +$$
$$\frac{(b-1)^n}{n!} f^{(n)}\left[a + \theta(b-a)\right]$$

b) If $f(x, y)=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$ when $xy\neq 0$, $-\frac{\pi}{2}\leq\tan^{-1}\left(\frac{x}{y}\right)\leq\frac{\pi}{2}$, and $f(x, 0)=f(0, y)=0$, then show that $\frac{\partial^2 f}{\partial x\partial y}(0, 0)\neq\frac{\partial^2 f}{\partial y\partial x}(0, 0)$.

c) Find the reduction formula for $\int \cos^m x\sin nx dx$, where $m$ and $n$ are positive integers and hence show that

$$I_{m,n}=\int_0^{\frac{\pi}{2}}\cos^m x\sin nx dx=\frac{1}{m+n}+\frac{m}{m+n}I_{m-1,n-1}.$$

d) Solve by method of variation of parameters $\frac{d^2y}{d^2x}+a^2y=\tan ax$.

e) Show that maximum value of is $(\frac{1}{x})^x$ is $e^{\frac{1}{e}}$.

f) If $\log y=\tan^{-1}x$ then show that

$$\left(1+x^2\right)\frac{d^{n+2}y}{dx^{n+2}}+(2nx+2x-1)\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\frac{d^n y}{dx^n}=0.$$

3. Answer any **two** questions: $10\times 2=20$

a) i) Show that the rectangle inscribed in a circle has maximum area when it is a square.

ii) Evaluate: $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$.



iii) Find a and b in order that

$$\lim_{x\to 0}\frac{a\sin 2x-b\sin x}{x^3}=1.$$

4+3+3

b) i) Determine a and b such that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1.$$

ii) Evaluate: $\mathrm{Lt}_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

iii) What is the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius a? 3+3+4

c) i) Show that

$$\frac{v-u}{1+v^2}<\tan^{-1}v-\tan^{-1}u<\frac{v-u}{1+u^2},$$

if $0<u<v$.

ii) Let $f:R\to R$ and $g:R\to R$ be two functions such that f(r) =g(r) = 0 where $r\in R$. Further consider $g'(r)\neq 0$. Then prove that $\lim_{x\to r}\frac{f(x)}{g(x)}=\frac{f'(r)}{g'(r)}$.