PRACTICE QUESTION PAPER — SET 1

PHY-M-T-12: Quantum Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Define a Hilbert space and write down the completeness (closure) relation for an orthonormal basis $\{|\phi_n\rangle\}$.

   (b) Prove that the eigenvalues of a Hermitian operator are always real.

   (c) Write the position operator $\hat{x}$ and momentum operator $\hat{p}_x$ in both position and momentum representations.

   (d) Define creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators for a one-dimensional harmonic oscillator and evaluate $[\hat{a}, \hat{a}^\dagger]$.

   (e) Write down the three Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ and verify that $\{\sigma_x, \sigma_y\} = 0$.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) State and prove the generalized uncertainty relation between two non-commuting Hermitian operators $\hat{A}$ and $\hat{B}$: $\Delta A\, \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$.

   (b) Obtain the transformation relation connecting the position-space wavefunction $\psi(x)$ and momentum-space wavefunction $\phi(p)$ via Fourier transform.

   (c) Compare the Schrödinger, Heisenberg, and Interaction pictures of quantum mechanics in terms of the time evolution of state vectors and operators.

   (d) Show that angular momentum $\vec{J}$ is the generator of infinitesimal spatial rotations, and construct the finite rotation operator $\hat{R}(\hat{n}, \theta)$.

   (e) Obtain the matrix representations of J_x, J_y, J_z, and J^2 for a spin-$1/2$ system ($j = 1/2$).

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Define projection operators $\hat{P}_i = |\phi_i\rangle\langle\phi_i|$ and prove their idempotency ($\hat{P}_i^2 = \hat{P}_i$) and orthogonality properties.

   (ii) Discuss the properties of the Parity operator $\hat{\Pi}$, determine its eigenvalues, and show that it commutes with the Hamiltonian of a symmetric potential $V(-x) = V(x)$.

   (b) (i) Express the Hamiltonian of a one-dimensional simple harmonic oscillator in terms of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ and deduce its energy eigenvalues algebraically.

   (ii) Define a coherent state $|\alpha\rangle$ as an eigenstate of the annihilation operator $\hat{a}$, normalize it, and show that it achieves the minimum uncertainty product $\Delta x\, \Delta p = \hbar/2$.

   (c) (i) Using the general angular momentum commutation relations $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$, derive the eigenvalues of J^2 and J_z.

   (ii) Explain the addition of two angular momenta $\vec{J} = \vec{J}_1 + \vec{J}_2$ and compute the Clebsch-Gordan coefficients for coupling two spin-$1/2$ particles ($j_1 = 1/2, j_2 = 1/2$).

PRACTICE QUESTION PAPER — SET 2

PHY-M-T-12: Quantum Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) State the Schwarz inequality for two state vectors $|\psi\rangle$ and $|\phi\rangle$ in an inner product space.

   (b) Show that the time evolution operator $\hat{U}(t, t_0)$ for a conservative system is unitary.

   (c) State the selection rules for dipole transitions in a one-dimensional harmonic oscillator using ladder operators.

   (d) Define Euler angles (α, β, γ) and write down the general rotation operator $\hat{D}(\alpha, \beta, \gamma)$.

   (e) Prove the Pauli matrix identity $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = (\vec{A} \cdot \vec{B})\hat{I} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B})$.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Discuss the concept of dual space (bra space) in Dirac notation and explain the Riesz representation theorem for linear functionals.

   (b) Derive the equation of continuity $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$ from the time-dependent Schrödinger equation and interpret the conservation of probability.

   (c) Derive the Heisenberg equation of motion for an operator $\hat{A}_H(t)$ and obtain Ehrenfest's theorem for expectation values.

   (d) Obtain the ground-state and n-th excited-state wavefunctions of a one-dimensional harmonic oscillator in position space using the annihilation and creation operators.

   (e) Discuss the properties of Spherical Harmonics $Y_l^m(\theta, \phi)$, including their orthonormality and behavior under parity transformation.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Explain how a change of basis is performed in a finite-dimensional vector space using a unitary transformation matrix S, and show how operator matrices transform under it.

   (ii) Prove that two Hermitian operators $\hat{A}$ and $\hat{B}$ share a complete set of simultaneous eigenstates if and only if $[\hat{A}, \hat{B}] = 0$.

   (b) (i) Formulate the Interaction (Dirac) picture of quantum mechanics when the Hamiltonian is split as $\hat{H} = \hat{H}_0 + \hat{V}(t)$, and derive the equations of motion for both state vectors and operators.

   (ii) Apply the Heisenberg picture to solve for the time evolution of position $\hat{x}_H(t)$ and momentum $\hat{p}_H(t)$ of a one-dimensional harmonic oscillator.

   (c) (i) Study the dynamics of a spin-$1/2$ particle placed in a uniform static magnetic field $\vec{B} = B_0\hat{k}$ (Larmor spin precession) using spinor algebra.

   (ii) State the triangle inequality for the addition of two angular momenta j_1 and j_2, and calculate the Clebsch-Gordan coefficients for $j_1 = 1, j_2 = 1/2$.

PRACTICE QUESTION PAPER — SET 3

PHY-M-T-12: Quantum Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) What are square-integrable functions (L^2 space)? Why are physical bound-state wavefunctions required to be square-integrable?

   (b) Evaluate the commutator $[\hat{x}^n, \hat{p}_x]$ for any positive integer n.

   (c) Define stationary states for a time-independent potential and show that the probability density $|\psi(\vec{r}, t)|^2$ is independent of time.

   (d) Show that a coherent state $|\alpha\rangle$ can be expanded in the number-state basis $\{|n\rangle\}$ as $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$.

   (e) Write down the differential representation of orbital angular momentum operators L_x, L_y, L_z, and L^2 in spherical polar coordinates (r, θ, ϕ).

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) State the fundamental postulates of quantum mechanics regarding state vectors, observables, measurement outcomes, and time evolution.

   (b) Explain the matrix representation of kets, bras, and linear operators in a discrete orthonormal basis, and show that $\langle\phi|\hat{A}|\psi\rangle = \Phi^\dagger A \Psi$.

   (c) Prove that the spatial translation operator $\hat{T}(\vec{a}) = \exp(-i\vec{a} \cdot \hat{\vec{p}}/\hbar)$ shifts position eigenstates as $\hat{T}(\vec{a})|\vec{r}\rangle = |\vec{r} + \vec{a}\rangle$.

   (d) Construct the rotation matrix $d^{(1/2)}_{m'm}(\beta)$ for a spin-$1/2$ system rotated by an angle β about the y-axis.

   (e) Derive the matrix elements of the raising and lowering operators J_+ and J_- in the simultaneous eigenbasis $|j, m\rangle$ of J^2 and J_z.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Discuss continuous bases $\{|x\rangle\}$ and $\{|p\rangle\}$, their Dirac delta normalization, completeness relations, and the inner product $\langle x|p\rangle$.

   (ii) Analyze the time evolution of a one-dimensional Gaussian wave packet in free space and demonstrate its spatial spreading over time.

   (b) (i) Using operator algebra, calculate the expectation values $\langle n|\hat{x}^2|n\rangle$ and $\langle n|\hat{p}^2|n\rangle$ in the n-th energy eigenstate of a harmonic oscillator and verify the virial theorem.

   (ii) Compute the time evolution of a coherent state $|\alpha(t)\rangle$ under the harmonic oscillator Hamiltonian and show that it remains a coherent state at all times.

   (c) (i) Establish the connection between the Wigner D-matrices $D^{(l)}_{m'm}(\alpha, \beta, \gamma)$ and Spherical Harmonics $Y_l^m(\theta, \phi)$.

   (ii) Explain the difference between the uncoupled representation $|j_1, j_2; m_1, m_2\rangle$ and the coupled representation $|j_1, j_2; j, m\rangle$, and state the orthogonality and selection rules for Clebsch-Gordan coefficients.

PRACTICE QUESTION PAPER — SET 4

PHY-M-T-12: Quantum Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Show that the eigenstates corresponding to distinct eigenvalues of a Hermitian operator are mutually orthogonal.

   (b) Define a function of an operator $f(\hat{A})$ and evaluate $e^{\hat{A}}\hat{B}e^{-\hat{A}}$ using the Baker-Campbell-Hausdorff lemma up to second-order commutators.

   (c) Write the expression for the time evolution operator $\hat{U}(t, t_0)$ when the Hamiltonian $\hat{H}(t)$ does not commute with itself at different times.

   (d) Verify that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$ for the harmonic oscillator number operator $\hat{N}$.

   (e) Why are half-integer values of orbital angular momentum l excluded from physical solutions of the spatial Schrödinger equation?

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Explain the principle of superposition of states and show how measurement probabilities are computed from expansion coefficients in an orthonormal basis.

   (b) Solve the time-independent Schrödinger equation for a free particle in momentum representation and transform the result back to position space.

   (c) Show that the expectation value of any observable $\langle \hat{A} \rangle$ is identical across the Schrödinger, Heisenberg, and Interaction pictures.

   (d) Discuss the geometrical interpretation of angular momentum quantization (vector model) and calculate the minimum angle between $\vec{L}$ and the z-axis for $l = 2$.

   (e) Construct the normalized singlet and triplet spin states for a system of two spin-$1/2$ particles and verify their eigenvalues under $\vec{S}^2$ and S_z.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Define the Hermitian adjoint $\hat{A}^\dagger$ of a linear operator $\hat{A}$ and prove that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

   (ii) Given a normalized state $|\psi\rangle = \frac{1}{\sqrt{6}}|u_1\rangle + \sqrt{\frac{2}{3}}|u_2\rangle + \frac{i}{\sqrt{6}}|u_3\rangle$, where $|u_n\rangle$ are orthonormal eigenstates of $\hat{H}$ with eigenvalues $E_n = n\hbar\omega$, find $\langle \hat{H} \rangle$ and ΔH.

   (b) (i) Derive the algebraic recurrence relations for the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ acting on energy eigenstates $|n\rangle$ of the harmonic oscillator.

   (ii) Show that two distinct coherent states $|\alpha\rangle$ and $|\beta\rangle$ are not orthogonal, and evaluate $|\langle\beta|\alpha\rangle|^2$.

   (c) (i) Obtain the matrix representations of J_x, J_y, and J_z for angular momentum $j = 1$.

   (ii) Using ladder operators $J_- = J_{1-} + J_{2-}$, explicitly determine the Clebsch-Gordan coefficients for the $j = 1/2$ states formed by coupling $j_1 = 1$ and $j_2 = 1/2$.