PRACTICE QUESTION PAPER — SET 1

PHY-M-T-6: Classical and Statistical Mechanics

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) What are cyclic (ignorable) coordinates? Show that the generalized momentum conjugate to a cyclic coordinate is conserved.

   (b) Define Poisson bracket of two dynamical variables u and v, and evaluate $[q_i, p_j]$.

   (c) Write down Boltzmann's entropy relation and explain the meaning of thermodynamic probability.

   (d) State Wien's displacement law and Rayleigh-Jeans law for blackbody radiation.

   (e) Define Fermi energy ε_F and write its expression for a three-dimensional free electron gas at $T = 0$ K.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) State D'Alembert's principle and derive Lagrange's equations of motion for a conservative holonomic system.

   (b) Obtain the Hamiltonian of a spherical pendulum and write down Hamilton's canonical equations of motion.

   (c) State and prove the Law of Equipartition of Energy in classical statistical mechanics. Mention its limitations regarding specific heats of solids and gases.

   (d) Deduce Wien's distribution law and Stefan-Boltzmann law from Planck's radiation formula.

   (e) Calculate the thermodynamic functions (internal energy, pressure, and entropy) of a photon gas using Bose-Einstein statistics.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Set up the Lagrangian for two identical coupled simple pendulums connected by a light spring of stiffness k, and determine their normal frequencies of oscillation.

   (ii) Discuss central force motion in a plane and show that the areal velocity remains constant.

   (b) (i) Derive the Sackur-Tetrode equation for the entropy of an ideal monatomic gas and explain how it resolves Gibbs' paradox.

   (ii) Analyze the thermodynamic properties of a two-level energy system and explain the concept of negative absolute temperature.

   (c) (i) Derive the Fermi-Dirac distribution law and obtain the expression for the specific heat of an electron gas in a metal at low temperatures ($T \ll T_F$).

   (ii) Explain the stability of white dwarf stars using the theory of a degenerate relativistic Fermi gas and discuss the Chandrasekhar mass limit.

PRACTICE QUESTION PAPER — SET 2

PHY-M-T-6: Classical and Statistical Mechanics

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Define a functional and write down the Euler-Lagrange condition for the extremization of the integral $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$.

   (b) Distinguish between macrostate and microstate with a suitable physical example.

   (c) What is the ultraviolet catastrophe in classical radiation theory?

   (d) State Kirchhoff's law of thermal radiation.

   (e) Mention two characteristic properties of liquid Helium-II below the λ-point.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Derive Euler-Lagrange equations of motion from Hamilton's principle of least action.

   (b) Show how homogeneity of time and isotropy of space lead to the conservation of total energy and total angular momentum respectively.

   (c) Derive the Maxwell-Boltzmann distribution law for the most probable distribution of particles among energy levels.

   (d) Write down Saha's ionization formula and explain its astrophysical significance in stellar spectra.

   (e) Obtain the Bose-Einstein distribution law and discuss the phenomenon of Bose-Einstein condensation.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Define Legendre transformation and derive Hamilton's canonical equations of motion from the Lagrangian.

   (ii) Using Poisson brackets, prove that the angular momentum components satisfy $[L_x, L_y] = L_z$ and $[L^2, L_z] = 0$.

   (b) (i) Define the single-particle partition function Z and express the internal energy U, Helmholtz free energy F, entropy S, and pressure P of an ideal gas in terms of Z.

   (ii) State Gibbs' paradox and explain why indistinguishable particles require dividing the phase-space volume by $N!$.

   (c) (i) Derive Planck's radiation formula using Bose-Einstein statistics applied to a photon gas inside a blackbody cavity.

   (ii) Calculate the average energy and Fermi pressure of a completely degenerate electron gas at absolute zero.

PRACTICE QUESTION PAPER — SET 3

PHY-M-T-6: Classical and Statistical Mechanics

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) What are generalized coordinates? State their advantages over Cartesian coordinates.

   (b) Prove that if a dynamical variable $u(q, p)$ does not explicitly depend on time and $[u, H] = 0$, then u is a constant of motion.

   (c) Define phase space and a statistical ensemble.

   (d) Write down the expression for the critical temperature T_c for Bose-Einstein condensation of an ideal Bose gas.

   (e) Compare the energy-momentum dispersion relation and equation of state for non-relativistic and ultra-relativistic degenerate Fermi gases.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Set up the Lagrangian and obtain the equation of motion for a simple pendulum of length l and bob mass m.

   (b) Reduce the two-body central force problem to an equivalent one-body problem and obtain the differential equation of the orbit.

   (c) Apply the Law of Equipartition of Energy to determine the molar specific heats C_V and C_P of monatomic and diatomic ideal gases.

   (d) Discuss the experimental spectral distribution of blackbody radiation and explain why neither Wien's law nor Rayleigh-Jeans law fits the entire spectrum.

   (e) Derive the thermodynamic functions (internal energy and pressure) of a strongly degenerate Bose gas below the condensation temperature T_c.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Construct the Hamiltonian for a one-dimensional harmonic oscillator and solve Hamilton's canonical equations to find $q(t)$ and $p(t)$.

   (ii) Prove the fundamental properties of Poisson brackets, including antisymmetry and bilinearity.

   (b) (i) Obtain the thermodynamic functions (mean energy, entropy, and specific heat) for a system of N non-interacting two-level particles with energies 0 and ε.

   (ii) Show under what condition a two-level system exhibits negative temperature and explain why a system at negative temperature is hotter than one at infinite positive temperature.

   (c) (i) Discuss the free electron model of metals using Fermi-Dirac statistics and show that the electronic contribution to specific heat is $C_{V,\text{el}} = \gamma T$.

   (ii) Derive the mass-radius relation $R \propto M^{-1/3}$ for a non-relativistic white dwarf star and explain why massive white dwarfs collapse beyond the Chandrasekhar limit.

PRACTICE QUESTION PAPER — SET 4

PHY-M-T-6: Classical and Statistical Mechanics

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) State Hamilton's variational principle in classical mechanics.

   (b) Write down the Lagrangian and Hamiltonian for a free particle moving in spherical polar coordinates (r, θ, ϕ).

   (c) State the fundamental postulate of equal a priori probabilities in statistical mechanics.

   (d) Write down Planck's radiation law in terms of wavelength λ and frequency ν.

   (e) Distinguish briefly between Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac statistics in terms of particle spin and symmetry of wavefunctions.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Derive Lagrange's equations of motion for a particle moving in a central force field $V(r)$ in plane polar coordinates (r, θ).

   (b) Using Poisson brackets, verify that the Runge-Lenz vector or angular momentum is conserved in a central potential $V(r)$.

   (c) Derive the Sackur-Tetrode expression for the entropy of a classical ideal gas using the canonical partition function.

   (d) Deduce Rayleigh-Jeans law from Planck's law in the long-wavelength limit and Stefan-Boltzmann law by integrating Planck's distribution over all frequencies.

   (e) Obtain the expression for the Fermi energy ε_F of an electron gas in a metal in terms of electron number density n.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Obtain the equations of motion of a spherical pendulum using the Lagrangian formulation and identify the conserved quantities.

   (ii) Explain how continuous symmetries of the Lagrangian lead to conservation laws (linear momentum, angular momentum, and energy).

   (b) (i) Discuss the properties of thermal radiation and derive Saha's thermal ionization equation, explaining the physical meaning of each term.

   (ii) Explain Bose-Einstein condensation and show that below T_c, the fraction of particles in the ground state is given by $\dfrac{N_0}{N} = 1 - \left(\dfrac{T}{T_c}\right)^{3/2}$.

   (c) (i) Derive the pressure and internal energy of a strongly degenerate Fermi gas at finite low temperature T ($k_B T \ll \varepsilon_F$) using Sommerfeld expansion.

   (ii) Discuss the equilibrium of a white dwarf star under gravitational contraction and electron degeneracy pressure, leading to the Chandrasekhar mass limit.