PRACTICE QUESTION PAPER — SET 1

PHY-M-T-11: Classical and Statistical Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) State Noether's theorem and give one physical example illustrating a conservation law.

   (b) Show that the transformation $Q = \ln\left(\dfrac{\sin p}{q}\right)$, $P = q\cot p$ is canonical.

   (c) What is the condition for a heavy symmetrical top with one point fixed to be a "sleeping top"?

   (d) Distinguish between a pure state and a mixed state in terms of the density matrix ρ.

   (e) Define the inversion temperature in a Joule-Thomson expansion and express it in terms of van der Waals constants a and b.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Derive Euler-Lagrange equations of motion from Hamilton's Principle of Least Action.

   (b) Prove that the fundamental Poisson brackets $[q_i, p_j] = \delta_{ij}$ are invariant under a canonical transformation.

   (c) Derive the quantum Liouville equation $\dfrac{\partial \rho}{\partial t} + \dfrac{1}{i\hbar}[\rho, H] = 0$ for the density operator ρ.

   (d) Obtain the expression for the second virial coefficient $B_2(T)$ for a real gas interacting via a central potential, and evaluate it for a van der Waals gas.

   (e) Explain Ehrenfest's classification of phase transitions with suitable examples of first-order and second-order transitions.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Set up the Hamilton-Jacobi equation for a one-dimensional simple harmonic oscillator and solve it to find the displacement $q(t)$.

   (ii) State Euler's theorem on the motion of a rigid body and derive Euler's dynamical equations of motion for a torque-free rigid body.

   (b) (i) Discuss the phenomenon of Bose-Einstein condensation in an ideal Bose gas and derive the expression for the critical condensation temperature T_c.

   (ii) Explain how a degenerate electron gas accounts for the stability of white dwarf stars and deduce the Chandrasekhar mass limit qualitatively.

   (c) (i) Formulate the Lagrangian density and Euler-Lagrange field equations for a continuous mechanical system by taking the continuum limit of a discrete linear chain.

   (ii) Present Landau's phenomenological theory of continuous phase transitions and derive the critical exponents α, β, γ, and δ within mean-field theory.

PRACTICE QUESTION PAPER — SET 2

PHY-M-T-11: Classical and Statistical Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Write down the Lagrangian and Hamiltonian for a charged particle moving in an electromagnetic field $(\vec{E}, \vec{B})$.

   (b) Define Lagrange brackets and state their relationship with Poisson brackets.

   (c) Explain Larmor precession of a magnetic dipole in a uniform magnetic field.

   (d) State Liouville's theorem in classical statistical mechanics and give its geometrical interpretation in phase space.

   (e) What is the order parameter in a ferromagnetic-to-paramagnetic phase transition? How does it behave near the Curie temperature T_c?

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Obtain the Hamiltonian from the Lagrangian using Legendre transformation, and derive Hamilton's canonical equations of motion.

   (b) Discuss infinitesimal contact transformations and show that the Hamiltonian H is the generator of time translations.

   (c) Construct the density matrix for an unpolarized and a partially polarized beam of spin-$\frac{1}{2}$ particles in terms of the polarization vector $\vec{P}$.

   (d) Derive Planck's radiation law for black-body radiation using Bose-Einstein statistics.

   (e) Define the stress-energy tensor $T^{\mu\nu}$ for a continuous field and prove the local conservation of energy and momentum.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Distinguish between Hamilton's principal function S and Hamilton's characteristic function W. Solve the Kepler problem using separation of variables in the Hamilton-Jacobi method.

   (ii) Discuss the precession and nutation of a heavy symmetrical top spinning with one point fixed under gravity.

   (b) (i) Derive the Fermi-Dirac distribution law and calculate the Fermi energy ε_F and average energy per electron of a degenerate Fermi gas at $T = 0$ K.

   (ii) Show that the specific heat of a degenerate Fermi gas at low temperatures ($T \ll T_F$) varies linearly with temperature T.

   (c) (i) Explain Gibbs' paradox in classical statistical mechanics and show how the correct Boltzmann counting resolves it.

   (ii) Discuss the Joule-Thomson throttling process for a van der Waals gas and show that the maximum inversion temperature is $T_{i,\text{max}} = \dfrac{2a}{Rb}$.

PRACTICE QUESTION PAPER — SET 3

PHY-M-T-11: Classical and Statistical Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Set up the Lagrangian for an Atwood machine and find the acceleration of the masses.

   (b) Write the transformation equations corresponding to the generating function $F_2(q, P, t) = \sum_i q_i P_i$.

   (c) State the integral invariants of Poincaré in phase space.

   (d) Show that $\text{Tr}(\rho^2) = 1$ for a pure state and $\text{Tr}(\rho^2) < 1$ for a mixed state.

   (e) Write the virial equation of state for a real gas and explain the physical meaning of the virial coefficients.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Show that the Euler-Lagrange equations are equivalent to Newton's second law of motion for a conservative mechanical system.

   (b) Derive the canonical transformation equations using the generating functions $F_1(q, Q, t)$ and $F_3(p, Q, t)$.

   (c) Find the density matrix for an electron of magnetic moment $\vec{\mu}$ placed in a uniform magnetic field $\vec{B} = B_0\hat{k}$ in thermal equilibrium at temperature T.

   (d) Derive the equation of state $PV = \frac{2}{3}U$ for an ideal non-relativistic quantum gas and compare it with an ultra-relativistic quantum gas ($PV = \frac{1}{3}U$).

   (e) Discuss the Hamiltonian formulation and Poisson brackets for continuous systems in momentum representation.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Discuss infinitesimal rotations of a rigid body and show that the angular velocity vector $\vec{\omega}$ is related to the rate of change of orientation.

   (ii) Explain the stability condition for a "fast top" and analyze the motion of an asymmetrical top qualitatively.

   (b) (i) Obtain the partition function for a grand canonical ensemble and derive the statistical definitions of internal energy, entropy, and pressure.

   (ii) Discuss lattice specific heat of solids using Einstein's and Debye's theories of phonons, highlighting the Debye T^3 law at low temperatures.

   (c) (i) Analyze continuous and discontinuous phase transitions using Landau's free energy expansion $F(\eta, T) = F_0 + a(T - T_c)\eta^2 + B\eta^4$.

   (ii) Show that within Landau theory, the entropy is continuous across T_c while the specific heat exhibits a finite discontinuity ΔC_P.

PRACTICE QUESTION PAPER — SET 4

PHY-M-T-11: Classical and Statistical Mechanics - II

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Explain when it is advantageous to use the Hamiltonian formalism over the Lagrangian formalism.

   (b) Prove Jacobi's identity for Poisson brackets: $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$.

   (c) Define action-angle variables in Hamilton-Jacobi theory.

   (d) What is a stationary ensemble? State the condition for a density matrix to represent a stationary state.

   (e) Define the critical exponents β and γ associated with the order parameter and generalized susceptibility near a critical point.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Use Noether's theorem to prove that homogeneity of space implies conservation of linear momentum and isotropy of space implies conservation of angular momentum.

   (b) Using Poisson brackets, show that if two quantities u and v are constants of motion, then their Poisson bracket $[u,v]$ is also a constant of motion (Poisson's theorem).

   (c) Obtain the density matrix in position representation for a free particle of mass m confined in a one-dimensional box of length L at temperature T.

   (d) Explain the properties of superfluidity in liquid Helium-II and its connection to Bose-Einstein condensation.

   (e) Discuss the liquid-gas phase transition using the van der Waals equation of state and locate the critical constants (P_c, V_c, T_c).

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Derive the Hamilton-Jacobi equation for Hamilton's principal function $S(q_i,\alpha_i,t)$ via a canonical transformation to constant generalized coordinates and momenta.

   (ii) Describe the working principle of a gyroscope and explain its gyroscopic stability.

   (b) (i) Express the number density n and energy density u of an ideal quantum gas in terms of Riemann's zeta function $\zeta(s)$ and Fermi/Bose integrals.

   (ii) Discuss the properties of high-temperature dense matter and the role of relativistic degenerate neutrons in neutron stars.

   (c) (i) Discuss the one-dimensional random walk problem and show that for a large number of steps N, the probability distribution approaches a Gaussian.

   (ii) Explain the divergence of generalized susceptibility χ_T and fluctuations of the order parameter near the critical point in mean-field theory.