B.SC. (HONOURS) PHYSICS • EXAMINATION PRACTICE BANK

Classical Mechanics of Point Particles

Four Model Question Sets • Lagrangian & Hamiltonian Formulations, Central Forces, Small Oscillations & Poisson Brackets

Set 1: Lagrangian Formulation, D'Alembert's Principle & Central Forces

SECTION A: SHORT ANSWER QUESTIONS (2 MARKS EACH)

1. What is meant by a functional in calculus of variations? Give one physical example.
2. Distinguish between holonomic and non-holonomic constraints with suitable examples.
3. State D'Alembert's principle and explain the physical significance of the term containing the virtual work of constraint forces.
4. What are generalized coordinates? Write down the relation between generalized velocities and Cartesian velocities for a system of N particles.
5. Write down the Lagrangian for a charged particle moving in an electromagnetic field described by scalar potential φ and vector potential $\mathbf{A}$.
6. State Kepler's Second Law of planetary motion and explain how it relates to the conservation of angular momentum in a central force field.

SECTION B: LONG ANSWER & DERIVATIONS (8 MARKS EACH)

1. **Derivation of Euler–Lagrange Equations:**
   - State Hamilton's principle of least action.
   - Using calculus of variations, derive the Euler–Lagrange equations of motion from Hamilton's principle for a conservative, holonomic system.
2. **Applications of Lagrangian Mechanics:**
   - Set up the Lagrangian for a **simple pendulum** of mass m and length l, and derive its equation of motion. Show that for small oscillations, it reduces to a simple harmonic oscillator.
   - Consider a particle of mass m constrained to move on the surface of a sphere of radius R under gravity. Find the Lagrangian using spherical polar coordinates (θ, φ) and obtain the equations of motion.
3. **Motion Under Central Force:**
   - Reduce the two-body central force problem into an equivalent one-body problem using reduced mass μ.
   - Write the equation of motion for a particle in a central force field and derive Binet's equation for the orbit:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{m}{L^2u^2}\,F(1/u)\,, \quad \text{where } u = 1/r.$$

Set 2: Small Oscillations, Coupled Systems & Cyclic Coordinates

SECTION A: SHORT ANSWER QUESTIONS (2 MARKS EACH)

1. Define a cyclic (or ignorable) coordinate. What conservation law is associated with a cyclic coordinate?
2. What is the physical significance of generalized momentum p_j conjugate to a generalized coordinate q_j?
3. Define the Lagrangian function L. Under what conditions does the total mechanical energy E equal the total energy function h defined by Jacobi's energy integral?
4. What do you understand by normal coordinates and normal modes of vibration in a coupled system?
5. Write down the potential and kinetic energy matrices (or terms) for two coupled harmonic oscillators.
6. State Noether's Theorem in brief relating spatial translation symmetry to a conservation law.

SECTION B: LONG ANSWER & DERIVATIONS (8 MARKS EACH)

1. **Cyclic Coordinates & Symmetries:**
   - Discuss how the homogeneity of space, isotropy of space, and homogeneity of time lead to the conservation of linear momentum, angular momentum, and total energy, respectively, using Lagrangian formalism.
   - Prove that if the Lagrangian does not explicitly contain a particular generalized coordinate q_j (cyclic coordinate), the corresponding canonical momentum p_j remains a constant of motion.
2. **Coupled Oscillators:**
   - Two identical particles of mass m are connected by a spring of force constant k and attached to fixed walls by two other identical springs of force constant k. Set up the Lagrangian, find the secular equation, and solve for the normal frequencies of vibration.
3. **Spherical Pendulum & Advanced Applications:**
   - Obtain the Lagrangian and equations of motion for a **spherical pendulum**. Identify the cyclic coordinates, find the constants of motion, and reduce the problem to a 1D effective potential motion in r or θ.

Set 3: Hamiltonian Formulation & Legendre Transformations

SECTION A: SHORT ANSWER QUESTIONS (2 MARKS EACH)

1. What is a Legendre transformation? State its primary utility in transitioning from Lagrangian to Hamiltonian mechanics.
2. Define the Hamiltonian H of a system. How is it related to the Lagrangian L through Legendre transformation?
3. Write Hamilton's canonical equations of motion for a system with n degrees of freedom. Why are they called first-order differential equations?
4. Define Phase Space. What is the dimension of the phase space for a system of 3 particles moving in 3-dimensional space?
5. Show that for a conservative system with velocity-independent potentials, the Hamiltonian represents the total mechanical energy of the system ($H = T + V$).
6. What are Hamilton's equations of motion for a charged particle moving in an electromagnetic field?

SECTION B: LONG ANSWER & DERIVATIONS (8 MARKS EACH)

1. **Hamiltonian Formalism & Equations:**
   - Using Legendre transformations, formally derive Hamilton's canonical equations of motion from the Lagrangian function.
   - Apply Hamilton's equations to find the equations of motion for:
     i. A one-dimensional harmonic oscillator.
     ii. A particle of mass m moving freely in a uniform gravitational field.
2. **Phase Space & Liouville's Theorem:**
   - Explain the concept of phase space and phase trajectories. State and explain Liouville's theorem in classical statistical mechanics.
3. **Comparison of Formalisms:**
   - Compare Newtonian, Lagrangian, and Hamiltonian mechanics in terms of:
     i. Choice of variables (coordinates vs. velocities vs. momenta),
     ii. Handling of constraint forces,
     iii. Order of differential equations, and
     iv. Suitability for complex systems and quantum mechanical transition.

Set 4: Poisson Brackets & Canonical Transformations

SECTION A: SHORT ANSWER QUESTIONS (2 MARKS EACH)

1. Define the Poisson bracket of two dynamical variables u and v in terms of canonical coordinates and momenta.
2. State the fundamental Poisson bracket relations among the canonical coordinates q_i and momenta p_j.
3. What is meant by an integral of motion? How is the total time derivative of a dynamical variable $A(q, p, t)$ expressed using Poisson brackets?
4. State Poisson's Theorem regarding the Poisson bracket of two constants of motion.
5. Define a canonical transformation. What is the primary condition for a transformation $(q, p) \rightarrow (Q, P)$ to be canonical?
6. Show that the Poisson bracket of two canonical variables is invariant under canonical transformations.

SECTION B: LONG ANSWER & DERIVATIONS (8 MARKS EACH)

1. **Properties of Poisson Brackets:**
   - Prove the following fundamental properties of Poisson brackets:
     i. Antisymmetry: $[u, v] = -[v, u]$
     ii. Linearity: $[cu + dw, v] = c[u, v] + d[w, v]$
     iii. Jacobi's Identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$
2. **Equations of Motion & Conservation Laws in Poisson Bracket Form:**
   - Express Hamilton's equations of motion using Poisson brackets.
   - Prove that if a dynamical variable A does not explicitly depend on time (t) and its Poisson bracket with the Hamiltonian vanishes ($[A, H] = 0$), then A is a conserved quantity (constant of motion).
3. **Advanced Problem Solving using Poisson Framework:**
   - Evaluate the Poisson brackets:
     i. $[q_i, p_j]$
     ii. $[L_x, L_y]$, where L_x, L_y are components of angular momentum. Hence, deduce $[L_x, L_y] = L_z$.
   - Use Poisson brackets to show that the total angular momentum squared $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with any single component of angular momentum (e.g., $[L^2, L_z] = 0$).