PRACTICE QUESTION PAPER — SET 1

PHY-M-T-1: Mathematical Physics-I

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Find the integrating factor of the differential equation: $x\dfrac{dy}{dx} + y = x^3y^6$.

   (b) State the Wronskian condition for two solutions of a second-order linear homogeneous differential equation to be linearly independent.

   (c) Calculate the directional derivative of $\phi = x^2yz$ at the point $(1, 2, -1)$ in the direction of the vector $\vec{i} + 2\vec{j} - 2\vec{k}$.

   (d) Define a unitary matrix. If U is unitary, show that $|\det U| = 1$.

   (e) Evaluate the integral: $\displaystyle\int_{-\infty}^{\infty} \delta(2x - 4)x^2\, dx$.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Expand $f(x) = \ln(1 + x)$ up to the third-order term using Taylor's theorem about $x = 0$, and state the condition of validity.

   (b) Find the particular integral (P.I.) of the differential equation: $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{3x}$.

   (c) Using Green's theorem in a plane, evaluate $\displaystyle\oint_C \left[(x^2 - y^2)\, dx + 2xy\, dy\right]$, where C is the boundary of the region enclosed by $y = x$ and $y = x^2$.

   (d) Derive the expression for the gradient operator $\nabla\phi$ in the cylindrical coordinate system (r, θ, z).

   (e) Find the eigenvalues and the corresponding eigenvectors of the matrix: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Verify Gauss's divergence theorem for the vector field $\vec{A} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

   (ii) Prove that the curl of the gradient of a scalar field is identically zero ($\nabla \times \nabla\phi = 0$).

   (b) (i) State and verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Hence find A^{-1}.

   (ii) Diagonalize the symmetric matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

   (c) (i) Find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$ using the method of Lagrange Multipliers.

   (ii) Define the Dirac delta function $\delta(x)$ as the limit of a Gaussian distribution. State two of its key scaling properties.

PRACTICE QUESTION PAPER — SET 2

PHY-M-T-1: Mathematical Physics-I

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Using the binomial approximation, approximate $(1 + 0.02)^{-3}$ up to second-order terms.

   (b) Find the Wronskian of the functions $y_1 = e^{mx}$ and $y_2 = e^{nx}$ ($m \neq n$).

   (c) Write down the volume element dV in spherical polar coordinates (r, θ, ϕ).

   (d) Show that the eigenvalues of a Hermitian matrix are always real.

   (e) Write down the probability mass function and mean for a Poisson distribution with parameter μ.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Solve the first-order differential equation: $\dfrac{dy}{dx} + y \cot x = \sin x$.

   (b) Find the general solution of the second-order homogeneous differential equation: $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$.

   (c) Find the work done by the force field $\vec{F} = (2x + y^2)\vec{i} + (3y - 4x)\vec{j}$ along the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

   (d) Derive the expression for the Laplacian operator ∇^2 in Cartesian coordinates.

   (e) If a random variable X follows a Gaussian (normal) distribution with mean μ and variance σ^2, write its probability density function and sketch its general shape.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) State Stokes' theorem. Verify it for $\vec{A} = y^2\vec{i} + xy\vec{j} - xz\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = R^2$ ($z \geq 0$).

   (ii) Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y\vec{i} + x\vec{j}$ along the straight line from $(0, 0)$ to $(1, 1)$.

   (b) (i) Find the inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ using matrix methods.

   (ii) Define a skew-Hermitian matrix. Show that the trace of a skew-Hermitian matrix is either purely imaginary or zero.

   (c) (i) Use the method of Lagrange multipliers to find the dimensions of a rectangular box of maximum volume with a given surface area S.

   (ii) Evaluate the integral: $\displaystyle\int_{-\infty}^{\infty} (3x^2 - 2x + 1)\delta(x - 3)\, dx$.

PRACTICE QUESTION PAPER — SET 3

PHY-M-T-1: Mathematical Physics-I

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) Test the exactness of the differential equation: $(2xy + 3)\,dx + (x^2 - 4)\,dy = 0$.

   (b) Write down the statement of the Existence and Uniqueness Theorem for second-order initial value problems.

   (c) Prove that $\nabla \cdot (\nabla \times \vec{A}) = 0$ for any twice-differentiable vector field $\vec{A}$.

   (d) Define an orthogonal matrix. If A is orthogonal, what can you say about A^{-1} and $\det A$?

   (e) Explain the representation of the Dirac delta function as the limit of a rectangular function.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Use Taylor series expansion to evaluate $\sqrt{1.04}$ up to three decimal places.

   (b) Find the particular integral of $\dfrac{d^2y}{dx^2} + y = \sin(2x)$.

   (c) Compute the Jacobian $J\left(\dfrac{x, y}{u, v}\right)$ for the transformation $x = u\cos v$, $y = u\sin v$.

   (d) Using orthogonal curvilinear coordinates, write down the scale factors h_1, h_2, h_3 for spherical polar coordinates, and hence find the line element ds^2.

   (e) State the properties of a binomial distribution and derive its mean.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) State Green's theorem in a plane. Use it to find the area of an ellipse with semi-axes a and b.

   (ii) Show that $\nabla^2\left(\dfrac{1}{r}\right) = 0$ for $r \neq 0$, where $r = \sqrt{x^2 + y^2 + z^2}$.

   (b) (i) Find the eigenvalues and eigenvectors of the real symmetric matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

   (ii) Diagonalize the matrix A found above and write down the corresponding similarity transformation matrix.

   (c) (i) Find the stationary points of the function $f(x, y) = x^3 + y^3 - 3axy$ and examine their nature (maxima, minima, or saddle points).

   (ii) Solve the system of coupled linear ordinary differential equations:

$$\frac{dx}{dt} = 2x - 3y, \qquad \frac{dy}{dt} = y - 2x$$

PRACTICE QUESTION PAPER — SET 4

PHY-M-T-1: Mathematical Physics-I

Full Marks: 40 **Time: 2 Hours**

Answer all questions. The figures in the margin indicate full marks. Symbols have their usual meanings.

1. **Answer any four questions:** **[2 × 4 = 8]**

   (a) What is an exact differential? Give one physical or mathematical example.

   (b) Find the eigenvalues of the diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

   (c) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

   (d) Define symmetric and skew-symmetric matrices with examples.

   (e) Evaluate the expression: $\int_{-2}^{2} (x^3 + 3x)\delta'(x)\, dx$.

2. **Answer any four questions:** **[4 × 4 = 16]**

   (a) Solve the linear differential equation with constant coefficients: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^{2x}$.

   (b) Solve the differential equation $(y^2 + y)\, dx - x\, dy = 0$ by finding a suitable integrating factor.

   (c) Derive the expression for the curl of a vector field in orthogonal curvilinear coordinates.

   (d) Prove the vector identity: $\nabla \times (\phi \vec{A}) = \nabla\phi \times \vec{A} + \phi(\nabla \times \vec{A})$.

   (e) Given a continuous random variable with a Gaussian distribution, write its variance and explain the physical significance of standard deviation.

3. **Answer any two questions:** **[8 × 2 = 16]**

   (a) (i) Verify Gauss's divergence theorem for $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$ over the cuboid bounded by $x = 0, x = 2, y = 0, y = 1, z = 0, z = 3$.

   (ii) Express the gradient and divergence operators in spherical polar coordinates.

   (b) (i) State and prove that the trace of a square matrix is invariant under a similarity transformation.

   (ii) Find the eigenvalues and verify Cayley-Hamilton theorem for the matrix: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

   (c) (i) Find the shortest distance from the origin to the plane $x + 2y + 3z = 13$ using the method of Lagrange multipliers.

   (ii) Discuss the properties of the Dirac delta function $\delta(x)$, specifically showing that $\delta(-x) = \delta(x)$ and $\delta(ax) = \frac{1}{|a|}\delta(x)$.