

- ii) Expand in Taylor series $f(z) = \frac{1}{1-z-z^2}$ about $z = 0$. 6+4
- c) i) State and prove the fundamental theorem of algebra.
- ii) State Cauchy's integral formula. Can we apply this formula to evaluate $\int_{\gamma} \frac{\sin \frac{1}{z}}{z} dz$, where $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$? Give reason in support of your answer. 5+5
- d) i) Let $f(z) = f(x+iy) = u(x, y) + iv(x, y)$ be defined and continuous in some neighbourhood of a point $z = z_0$. Then show that both $u(x, y)$ and $v(x, y)$ are continuous at z_0 .
- ii) Prove that if f is continuous at $z_0 \in D$, then for every sequence $\{z_n\}$ in D , converging to z_0 , $f(z_n)$ converges to $f(z_0)$. D is an open connected set in C . 5+5

U.G. 5th Semester Examination - 2025

MATHEMATICS

[PROGRAMME]

Discipline Specific Elective (DSE)

Course Code : MATH-G-DSE-T-1(A&B)

[CBCS]

[New Syllabus]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

Answer all the question from Selected Option.

OPTION-A

MATH-G-DSE-T-1A

(Group Theory and Linear Algebra)

1. Answer any **ten** questions: 2×10=20
 - i) If every element of a group is its own inverse, then show that the group is abelian.
 - ii) Find the order of [6] in the group \mathbb{Z}_{14} .
 - iii) Find the index of the subgroup $H = \{e, a^2\}$ in the group $G = \{e, a, a^2, a^3, a^4, a^5\}$, where G is cyclic.

- iv) Is the symmetric group S_3 abelian? Justify.
- v) If the vectors $(0, 1, a)$, $(1, a, 1)$, $(a, 1, 0)$ of the vector space \mathbb{R}^3 be linearly dependent, then find the value of 'a'.
- vi) Verify whether the set $\{1+x, 1-x, x^2\}$ is linearly independent in the vector space of polynomials of degree at most 2.
- vii) If x be an eigenvalue of an orthogonal matrix A , then show that $\frac{1}{x}$ is also an eigenvalue of A .
- viii) A transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x, 0)$. Check whether it is linear or not.
- ix) Find all normal subgroups of the group \mathbb{Z}_4 .
- x) Check whether $\{(x, x+1) : x \in \mathbb{R}\}$ is a subspace of \mathbb{R}^2 or not.
- xi) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3 + 2x - 3$. Find $f^{-1}(3)$.
- xii) Define basis of a vector space with an example.
- xiii) Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 5 & 4 & 2 \end{pmatrix}.$$

- xiv) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

- xv) Find $k \in \mathbb{R}$ so that the set $S = \{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$ is linearly dependent in \mathbb{R}^3 .

2. Answer any **four** questions: 5×4=20

- i) Let H be the set of all matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}.$$

Prove that H is a subgroup of $GL_2(\mathbb{R})$.

- ii) Show that the vectors $\{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$ form a basis of the vector space \mathbb{R}^3 over \mathbb{R} .

- iii) Prove that the subset $H = \{i, (1234), (1432), (13)(24)\}$ of S_4 forms a cyclic subgroup of the symmetric group S_4 . Find the generators of H .

- iv) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $T(x, y, z) = (x-y, x+2y, y+3z)$, $(x, y, z) \in \mathbb{R}^3$. Show that $\text{Ker } T = \{\theta\}$.

- v) For any two elements a and b of a group G , show that $(ab)^2 = a^2b^2$ if and only if $ab = ba$ in G .

- vi) Show that Subgroup of a Cyclic group is Cyclic.

3. Answer any **two** questions: 10×2=20

- i) a) Prove that every subgroup of a cyclic group is cyclic.

- b) Show that the planes passing through the origin is a proper subspace of \mathbb{R}^3 . 5+5
- ii) a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$.
- b) Let S be the subspace of \mathbb{R}^3 defined by $S = \{(x, y, z) \in \mathbb{R}^3 \mid y = z = 0\}$. Show that S is a subspace of \mathbb{R}^3 . 5+5
- iii) a) Find the dimension of the subspace S of \mathbb{R}^3 defined by $S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y = z, 2x + 3z = y\}$.
- b) Show that the set of vectors $S = \{(1, 2, 0), (2, 1, 3), (1, 1, 1), (2, 3, 1)\}$ is linearly dependent in \mathbb{R}^3 . Find a linearly independent subset S_1 of S such that $L(S_1) = L(S)$.
- iv) a) On the set of integers Z , the binary operation '*' is defined by $a * b = a + b - 2$ for all $a, b \in Z$. Show that $(Z, *)$ is a group.
- b) Define order of a group. Show that every group of prime order is cyclic. 6+(1+3)

OPTION-B
MATH-G-DSE-T-1B
(Complex Analysis)

1. Answer any **ten** questions: 2×10=20
- a) Show that if $f(z)$ is analytic at a point $z = z_0$, then it is bounded in a suitable neighbourhood of z_0 .
- b) If $\text{Arg}(z+3) = \frac{\pi}{3}$ then find the least value of $|z|$.
- c) By the definition of limit show that $\lim_{z \rightarrow 1} |z - 2| = 1$.
- d) If a function f is continuous in $D = \{z : |z| \leq 1\}$, then show that it is bounded there.
- e) Find the radius of convergence of $1 + z + z^2 + z^3 + \dots$.
- f) If $z = \cos(\sin \frac{\pi}{2}) + i \sin(\sin \frac{\pi}{2})$ then find the least value of $|\text{Im } z|$.
- g) If $f + g$ is analytic at a point $z = z_0$, can we conclude that each of the functions f and g is also analytic at z_0 ?
- h) Evaluate $\int_{\gamma} \frac{1}{z} dz$, where $\gamma = \{z : |z| = 1\}$.
- i) If $f(z) = |\text{Re}(z) \text{Im}(z)|$ for $z \in C$. Check the continuity of f at $z = 0$.

j) Discuss the continuity of the function $f(z) = \frac{\sin z}{z}$, if $z \neq 0$ and $f(z) = 1$ if $z = 0$ at the point $z = 0$.

k) If $|z - (4 - 3i)| = 2$, then find the greatest value of $|z|$.

l) Show that $\int_{\gamma} z dz = 0$, where $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$.

m) Evaluate $\int_{|z|=r} |z-r| |dz|$.

n) Check whether $f(z) = x^2 + iy^2$ is analytic at the origin.

o) Which of the functions $\sin z$ and $\sin|z|$ are bounded and why? Give arguments.

2. Answer any **four** questions: $5 \times 4 = 20$

a) State Cauchy's integral formula. Use this

formula to evaluate $\int_{|z|=1} \frac{\sin z^2}{z} dz$.

b) If f is analytic at a point z_0 and g is analytic at $f(z_0)$, then show that $g(f(z))$ is analytic at z_0 .

c) Prove the following inequality:

$$\left| \int_{\gamma} f(z) dz \right| \leq ML, \text{ where } L = \text{the length of the}$$

curve γ and $M = \max\{|f(\gamma(t))| : t \in [a, b]\}$.

d) Find the radius of convergence of $1 + az + a^2 z^2 + a^3 z^3 + \dots$, and from it find the value of a so that the above power series converges at $z = 1 - i$.

e) Show that $f(z) = (\bar{z} + 2i)^2 - 2i$ is nowhere differentiable.

f) Evaluate the following line integral: $\int_{\gamma} x dz$, where γ is the straight line segment from 0 to $a + ib, a, b \in \mathbb{R}$.

3. Answer any **two** questions: $10 \times 2 = 20$

a) i) State Liouville's theorem. Use it to prove the fundamental theorem of abstract algebra.

ii) Show that $\left| \int_{|z|=1} e^z dz \right| \leq 2\pi e$. $5+5$

b) i) Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a series with radius of convergence R . If $r < R$ and

$M(r) = \max\{|f(z)| : |z - z_0| = r\}$, then

show that for each $k \in \mathbb{N}$, $|a_k| \leq \frac{M(r)}{r^k}$,

$k = 1, 2, 3, \dots$