U.G. 2nd Semester Examination - 2023

MATHEMATICS

[HONOURS]

Generic Elective Course (GE)

Course Code : MATH-H-GE-T-02

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Find the degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} - 2x\frac{dy}{dx} = sin\left(\frac{d^2y}{dx^2}\right).$$

b) Show that the substitution $z = logx$ transforms the equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0 \text{ into}$$

$$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 0.$$

c) Show that the differential equation of the family of circles of fixed radius r with centers on y axis is $(x^2 - r^2)\left(\frac{dy}{dx}\right)^2 + x^2 = 0.$

[Turn over]

d) Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$.

e) If f is differentiable at c and $f(c) \neq 0$, then show that $\frac{1}{f}$ is also dilferentiable at c.

f) Show with reason that the equation $x - \cos x = 0$ has a real root in $\left(0, \frac{\pi}{2}\right)$.

g) Show that $\lim_{x \to 0} \frac{1}{1+e^{\frac{1}{x}}}$ does not exist.

h) If the function $f(x)$satisfies the condition $f(x+y) = f(x)f(y)\ \forall x, y$ and $f(x) = 1 + xg(x)$ where $g(x) \to 1$as $x \to 0$. Then show that $f'(x) = f(x)$.

i) Explain why L' Hospital rule is not applicable on $\lim_{x \to 0} \frac{\sin x}{x}$.

j) Examine that Rolle's theorem is applicable on the function $f(x) = \begin{cases} 1 - x^2, x \leq 0 \\ \cos x, x > 0 \end{cases}$ in $\left[-1, \frac{\pi}{2}\right]$.

k) Find the singular solution of the differential equation $xp^2 = (x-a)^2$.

l) Evaluate: $\int_0^{\frac{\pi}{2}} cos^7 x dx$.

m) If $V = sin^{-1}\left(\sqrt{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}}\right)$, then show that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = -\frac{1}{12}tanV.$$

n) Explain whether the function $f(x) = [x]$ is continuous at $x = 1$ or not.

o) If $y_1 = 1 + x$ and $y_2 = e^x$ be two solutions of $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$ Find $P(x)$.

2. Answer any **four** questions: 5×4=20

a) If $u = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

b) Determine

$$\lim_{x\to\infty}[x - \sqrt[n]{(x-a_1)(x-a_2)\ldots\ldots\ldots(x-a_n)}].$$

c) Show that at $x = \frac{1}{4}$ the function $f(x) = \frac{1}{8}log\, x - bx + x^2, x > 0$, where $b \geq 0$ is a constant has neither maximum nor minimum.

d) If $y = x\log\left(\frac{x-1}{x+1}\right)$, prove that

$$y_n = (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

e) If $I_{m,n} = \int \sin^m x \cos^n x dx$, then show that

$$I_{m,n} = \frac{\sin^{m+1}x\cos^{n-1}x}{(m+n)} + \frac{(n-1)}{(m+n)} I_{m,n-2}.$$

f) Find the general and singular solution of differential equation:

$$(px^2 + y^2)(px + y) = (p+1)^2.$$

g) Solve the differential equation by the method of undetermined coefficients:

$$(D^2 + 2D + 2)y = x^2 + \sin x.$$

3. Answer any **two** questions: 10×2=20

a) i) If $y = (\sinh^{-1}x)^2$, prove that

$$(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0.$$

ii) Show that

$$x > \log(1+x) > x - \frac{1}{2}x^2;\ (x > 0).$$

5+5

b) i) Find the expansion of $(1+x)^n$ in a power series of x and indicate the range of validity of the expansion.

ii) If $f(x+y)=f(x)+f(y)$ for all x and y and $f(x)$ is continuous at $x=0$. Then show that $f(x)$ is continuous for all values of x. 5+5

c) i) If H be a homogeneous function in x, y, z of degree n then show that :

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right)=0,$$

if $u=(x^2+y^2+z^2)^{-\frac{(n+1)}{2}}$.

ii) If $\phi(x)$ be a polynomial in x and λ is a real number then prove that there exists a root of $\phi'(x)+\lambda\phi(x)=0$ between any pair of roots of $\phi(x)=0$. 5+5

d) i) Show that $x^{-\frac{1}{2}}cosx$ and $x^{-\frac{1}{2}}sinx$ be two linearly independent solutions of $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\frac{1}{4}\right)y=0$ and find the general solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\frac{1}{4}\right)y=x^{-\frac{3}{2}}.$$

ii) Solve the differential equation:

$$(2+x)^2\frac{d^2y}{dx^2}+(2+x)\frac{dy}{dx}+4y=2sin(2\log(2+x))$$

5+5