U.G. 2nd Semester Examination - 2023 MATHEMATICS

[HONOURS]

Course Code: MATH-H-CC-T-03

(Real Analysis)

Full Marks: 60 Time: $2\frac{1}{2}$ Hours The figures in the right-hand margin indicate marks. The symbols and notations have their usual meanings.

1. Answer any ten questions:

 $2 \times 1.0 = 20$

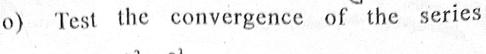
- Prove that the set of all open intervals having rational end points is a countable set.
 - b) Using the definition of a compact set, prove that a finite subset of \mathbb{R} is a compact set in \mathbb{R} .
- Let $A = \{x \in \mathbb{R} : x^2 5x + 6 < 0\}$. Find the *lub* and *glb* of A.
 - d) Show that the union of an infinite number of closed sets in \mathbb{R} is not, necessarily a closed set.
 - e) Prove or disprove $(A \cap B)' = A' \cap B'$ and $(A \cup B)' = A' \cup B'$, where A' is the derived set of the s set A.

[Turn over]

- f) If S be a non-empty compact subset of \mathbb{R} , then prove that sup S and inf S belong to S.
- g) Prove that $\lim r^n = 0, [r] < 1$.
- Three subsequences $\{x_{3n}\}, \{x_{3n-1}\}, \text{ and } \{x_{3n-2}\}$ of a sequence $\{x_n\}$ converge to the same limit x, then prove that the sequence $\{x_n\}$ is also converges to the same limit x.
 - i) Let $S \subseteq \mathbb{R}$ and x be a limit point of S. Then show that every neighbourhood of x contains infinitely many elements of S.
- Prove that a Cauchy sequence of real numbers is convergent.
 - k) Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in \mathbb{R} , then show that $\{x_ny_n\}$ is also a Cauchy sequence.
 - 1) Prove that a bounded sequence $\{x_n\}$ is convergent if and only if $\overline{\lim} x_n = \underline{\lim} x_n$
 - m). Prove that

$$\lim_{n\to\infty} \frac{1}{n} \{ (a+1)(a+2)...(a+n) \}^{\frac{1}{n}} = \frac{1}{e} \text{if } a > 0.$$

n) State and prove the necessary condition for the convergence of a series of positive terms. Verify whether it is also sufficient condition or not.



$$1 + \frac{1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \dots$$

2. Answer any four questions:

- a) Prove that dervied the set A' of a bounded set $A \subset \mathbb{R}$ is a compact set in \mathbb{R} .
- Find the derived set of the set $A = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}.$
- c) Using Archimedean property to show that the set \mathbb{Z} of intergers is neither bounded above nor bounded below.
- d) If A be a compact set in \mathbb{R} , then show that every infinite subset of A has a limit point in A.
- e) Prove that the set of real numbers \mathbb{R} is not countable.
- f) If $\lim_{n\to\infty} x_n = a$, then prove that $\lim_{n\to\infty} \frac{x_1 + x_2 + ... + x_n}{n} = a$. Is the converse true? Justify your answer.
- g) Using Gauss's test to show that the series $\left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right) + \dots \text{ converges for } p > 2 \text{ and diverges for } p \le 2.$

[Turn over]

3. Answer any two questions:

- a) i) Let $A \subset \mathbb{R}$ Prove that $(A^o)^c = \overline{A}^c$ and $(A)^c = (\overline{A}^c)^o$, where A^o, \overline{A}, A^c are the interior, closure, and complement of the set A respectively.
 - ii) Let A, B be two disjoint compact subsets of \mathbb{R} , then show that d(A, B) > 0, where $d(a,b) = \inf \{|a-b| : a \in A, b \in B\}$. 5+5
- b) i) State and prove Cantor's theorem on nested intervals in \mathbb{R} .
 - ii) Let $0 < y_1 < x_1$. Define

$$x_{n+1} = \frac{x_n + y_n}{2}$$
 and $y_{n+1} = \sqrt{x_n y_n}$.

Prove that the sequence $\{y_n\}$ is monotonic increasing and the sequence $\{x_n\}$ is monotonic decreasing but both the sequences converge to the same limit.

c) i) State Cauchy's condensation test and use it to discuss the convergence of the

series
$$\sum_{2}^{\infty} \frac{1}{n(\log n)^{p}}, p > 0.$$

(ii) Show that the series

$$\frac{1}{(1+a)^r} - \frac{1}{(2+a)^r} + \frac{r}{(3+a)^r} - \dots \text{ is}$$
absolutely con vergent for $p > 1$ and

conditionally convergent for 0 .

5+5

