U.G. 2nd Semester Examination - 2023

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-03

(Real Analysis)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any ten questions: 2×10=20

a) Prove that the set of all open intervals having rational end points is a countable set.

b) Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.

c) Let $A=\{x\in\mathbb{R}: x^2-5x+6<0\}$. Find the *lub* and *glb* of A.

d) Show that the union of an infinite number of closed sets in $\mathbb{R}$ is not, necessarily a closed set.

e) Prove or disprove $(A\cap B)'=A'\cap B'$ and $(A\cup B)'=A'\cup B'$, where A' is the derived set of the s set A.

[Turn over]

f) If S be a non-empty compact subset of $\mathbb{R}$, then prove that $\sup S$ and $\inf S$ belong to S.

g) Prove that $\lim r^n = 0, [r] < 1$.

h) Three subsequences $\{x_{3n}\}, \{x_{3n-1}\}$, and $\{x_{3n-2}\}$ of a sequence $\{x_n\}$ converge to the same limit x, then prove that the sequence $\{x_n\}$ is also converges to the same limit x.

i) Let $S \subseteq \mathbb{R}$ and x be a limit point of S. Then show that every neighbourhood of x contains infinitely many elements of S.

j) Prove that a Cauchy sequence of real numbers is convergent.

k) Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in $\mathbb{R}$, then show that $\{x_n y_n\}$ is also a Cauchy sequence.

l) Prove that a bounded sequence $\{x_n\}$ is convergent if and only if $\overline{\lim}\, x_n = \underline{\lim}\, x_n$

m). Prove that

$$\lim_{n\to\infty} \frac{1}{n}\{(a+1)(a+2)\ldots(a+n)\}^{\frac{1}{n}} = \frac{1}{e} \text{ if } a > 0.$$

n) State and prove the necessary condition for the convergence of a series of positive terms. Verify whether it is also sufficient condition or not.

o) Test the convergence of the series

$$1+\frac{1}{1!}+\frac{2^2}{2!}+\frac{3^3}{3!}+\ldots$$

2. Answer any **four** questions: $5\times4=20$

a) Prove that dervied the set A' of a bounded set $A\subset\mathbb{R}$ is a compact set in $\mathbb{R}$.

b) Find the derived set of the set

$$A=\left\{\frac{1}{m}+\frac{1}{n}: m,n\in\mathbb{N}\right\}.$$

c) Using Archimedean property to show that the set $\mathbb{Z}$ of intergers is neither bounded above nor bounded below.

d) If A be a compact set in $\mathbb{R}$, then show that every infinite subset of A has a limit point in A.

e) Prove that the set of real numbers $\mathbb{R}$ is not countable.

f) If $\lim_{n\to\infty} x_n = a$, then prove that

$$\lim_{n\to\infty}\frac{x_1+x_2+\ldots+x_n}{n}=a.$$ Is the converse true? Justify your answer.

g) Using Gauss's test to show that the series $\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)+\ldots$ converges for $p>2$ and diverges for $p\leq 2$.

3. Answer any **two** questions: 10×2=20

a) i) Let $A \subset \mathbb{R}$ Prove that $(A^o)^c = \overline{A^c}$ and $(A)^c = (\overline{A^c})^o$, where $A^o, \overline{A}, A^c$ are the interior, closure, and complement of the set A respectively.

ii) Let A, B be two disjoint compact subsets of $\mathbb{R}$, then show that $d(A, B) > 0$, where $d(a,b) = \inf\{|a-b| : a \in A, b \in B\}$. 5+5

b) i) State and prove Cantor's theorem on nested intervals in $\mathbb{R}$.

ii) Let $0 < y_1 < x_1$. Define

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n}.$$

Prove that the sequence $\{y_n\}$ is monotonic increasing and the sequence $\{x_n\}$ is monotonic decreasing but both the sequences converge to the same limit. 5+5

c) i) State Cauchy's condensation test and use it to discuss the convergence of the series $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}, p > 0$.

ii) Show that the series

$$\frac{1}{(1+a)^r} - \frac{1}{(2+a)^r} + \frac{r}{(3+a)^r} - \dots$$ is

absolutely con vergent for $p > 1$ and conditionally convergent for $0 < p \leq 1$.

5+5