U.G. 3rd Semester Examination - 2024

MATHEMATICS

[MAJOR]

Course Code : MATH-MT-03

(Real Analysis)

[NEP-2020]

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

   i) Show that if $f(x)$ is continuous on a compact set, then it is uniformly continuous there.

   ii) Does Rolle's theorem hold for $f(x) = 2 + (x-2)^{2/3}$ on [1,3] ? Justify your answer.

   iii) Show that the function,

   $$f(x)=\begin{cases} x^2 sin(1/x), & x\neq 0 \\ 0, & x=0 \end{cases}$$ is differentiable but not continuously differentiable at $x = 0$.

   iv) Find the radius of convergence of the power series $\sum n!x^n$.

   v) Prove that the set of all limit points of a given set is closed.

   vi) Prove that a differentiable function with a bounded derivative is Lipschitz continuous.

[Turn over]

vii) Is the sequence $x_n = n(-1)^n$ bounded? Justify your answer.

viii) Show that a sequence (x_n) satisfying $|x_{n+1} - x_n| \to 0$ need not be convergent.

ix) Show that the ratio test is inconclusive when

$$\lim\sup\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

x) Prove that every monotonic function on $[a, b]$ is Riemann integrable.

xi) Show that the alternating series test is not sufficient to guarantee the absolute convergence of a series.

xii) Prove that a bounded and monotonic sequence of partial sums must converge.

xiii) Show that the supremum of a continuous function on an open interval need not be attained.

xiv) Show that if a sequence of continuous functions $f_n(x)$ converges uniformly to $f(x)$, then $f(x)$ must be continuous.

xv) Define a convex function with an example.

2. Answer any **four** questions: 5×4=20

i) Define a Cauchy sequence with an example. Prove that a sequence (x_n) is Cauchy if and only if all of its subsequences are also Cauchy.

ii) Prove that Dirichlet function on $[0,1]$ is discontinuous everywhere.

iii) Define functions of bounded variation on $[a, b]$. Prove that a function is of bounded variation on $[a, b]$ if and only if it can be expressed as the difference of two monotone increasing functions on $[a, b]$.

iv) Prove that the function $f(x)$ is continuous at every irrational number but discontinuous at every rational number, where

$$f(x)=\begin{cases} 1/q, & x=p/q,\ p\in\mathbb{Z} \textit{ and } q\in\mathbb{N} \textit{ are coprime} \\ 0 & x \text{ is irrational} \end{cases}$$

v) Consider the sequence of functions $f_n(x)=(1-x)^n$ on $[0,1]$. I) Find pointwise limit function $f(x)$. II) Prove that $f_n(x)$ does not converge uniformly.

vi) If $f:[a,b]\to\mathbb{R}$ be differentiable, then show that f' satisfies the intermediate value property.

3. Answer any **two** questions: 10×2=20

i) a) Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

b) Provide an example of a compact metric space that is not sequentially compact, or prove that such an example is impossible.

5+5

ii) a) Let (x_n) be a bounded sequece in $\mathbb{R}$. Prove that it has convergent subsequence (Bolzano-Weierstrass theorem).

b) Give an example of a sequence that is bounded but has not limit points in $\mathbb{Q}$.

c) Prove that if $x_n \to L$ and $y_n \to M$, then $\lim\sup (x_n + y_n) = \lim\sup x_n + \lim\sup y_n$.

5+2+3

iii) a) Provide an example of a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$, and explain why.

b) Prove that $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous but it is not a function of bounded variation.

3+7

iv) a) Show that the sequence of functions $f_n(x) = \dfrac{x}{1+nx}$ on $[0,1]$ converges point-wise but not uniformly.

b) Prove that the improper integral, $\int_0^1 \frac{1}{x^p} dx$ converges if and only if $p > 1$.

5+5
