U.G. 4th Semester Examination - 2025

MATHEMATICS

[MAJOR]

Course Code : MATH-MJ-T-5

(Algebra-II)

[NEP-2020]

Full Marks : 60

Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.
The notations and symbols have their usual meanings.

GROUP-A

1. Answer any **ten** questions: $2\times10=20$

   a) Give an example of an infinite group in which every element has finite order.

   b) Prove that every proper subgroup of a group of order 6 is cyclic.

   c) Prove that center of a group G is a characteristic subgroup of G.

   d) Is the external direct product $\mathbb{Z}_6 \times \mathbb{Z}_4$ cyclic? Justify your answer.

   e) Define internal direct product of groups.

   f) Show that center of a group of order 65 is isomorphic to $\mathbb{Z}_{65}$.

   g) State Sylow's First Theorem.

   h) Show that the mapping $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=3^x$, $x\in\mathbb{R}$ is not a ring homomorphism.

   i) Show that $\frac{\mathbb{Z}}{3\mathbb{Z}}\simeq\frac{4\mathbb{Z}}{12\mathbb{Z}}$.

[Turn over]

j) Determine k so that $(1,3,1)$, $(2, k, 0)$ and $(0,4,1)$ are linearly dependent.

k) Is the mapping $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T((x,y)) = (x^2, y^2)\forall(x.y)\in\mathbb{R}^2$ a linear transformation? Justify your answer.

l) Let V be a vector space with a basis $\{e^{3t}, te^{3t}, t^2e^{3t}\}$ and $D:V\to V$ be defined by $Df(t) = \frac{d}{dt}f(t),\ f(t)\in V$. Find the matrix of D in the given basis.

m) Is the matrix $\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ diagonalisable? Justify your answer.

n) Use Cayley Hamilton theorem to compute A^{-1}, where $A = \begin{pmatrix}3 & -1\\ 2 & 4\end{pmatrix}$.

o) If $F(x) = \begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}x + \begin{pmatrix}0 & 1\\ 1 & 3\end{pmatrix}$, find *adj* $F(x)$ as a matric polynomial.

GROUP-B

2. Answer any **four** questions: $5\times4=20$

a) Let G and G' be two groups and let $\phi: G\to G'$ be an onto homomorphism. Prove that $G/Ker\phi \cong G'$.

b) Prove that for any group $G, |G/Z(G)| \neq 91$.

c) Examine whether the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a,b \in \mathbb{R}\right\}$ is a field.

d) Show that the set of vectors $S = \{(1,2,0), (3,-1,1), (4,1,1)\}$ is linearly dependent in $\mathbb{R}^3$. Apply Deletion theorem to find a proper subset S_1 of S such that $L(S_1) = L(S)$.

e) Find a basis for the row space of the matrix

$$\begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & 0 & 6 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

f) Let V and W be vector spaces over a field F with V finite dimensional. If $T: V \to W$ is a linear mapping, then prove that $\dim Ker\ T + \dim Im\ T = \dim V$.

GROUP-C

3. Answer any **two** questions: $10 \times 2 = 20$

a) i) For a group G, prove that $Inn(G)$ is a normal subgroup of $Aut(G)$. 3

ii) Examine whether the ring of matrices $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a,b \in \mathbb{Z}\right\}$ contains the unity. 2

iii) Extend the set $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of $\mathbb{R}^4$. 5

b) i) If G be a finite abelian group and d be a positive divisor of $o(G)$, then prove that G has a subgroup of order d. 5

ii) Show that every subgroup of a cyclic group is a characteristic subgroup. 5

c) i) Prove that intersection of two ideals of a ring R is an ideal of R. 2

ii) Show that the mapping $f : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{3}]$ defined by $f(a + b\sqrt{2}) = a + b\sqrt{2}$ is a group homomorphism but not a ring homomorphism. 3

iii) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, generated by the linearly independent set $\{(1,1,0,1), (1,1,0,0), (0,1,0,1)\}$. 5

d) i) Diagonalise the matrix $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$. 5

ii) If a group G be the internal direct product of its subgroups H and K, then prove that $G \cong H \times K$. 5
