B.Sc. Mathematics (Honours)

Class Notes: Sequences and Series in Real Analysis

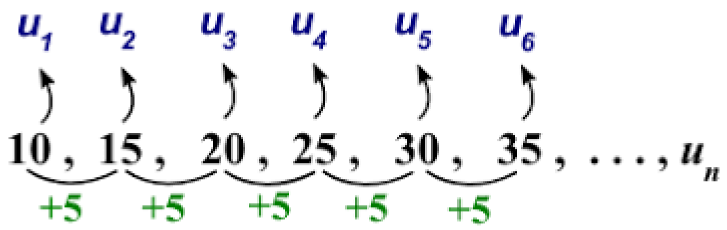

Table of Contents:

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Sequences and Series in Real Analysis

1. Introduction:

- In real analysis, sequences and series play a fundamental role in understanding the behaviour of functions and their limits.
- A sequence is an ordered list of numbers, denoted as $\{a_n\}_{n=1}^{\infty}$, where n is a positive integer. Each term in the sequence is represented by a subscript n.
- A series is the sum of the terms in a sequence, denoted as $\sum_{n=1}^{\infty} a_n$,

  where the symbol $\sum$ represents the sum.

2. Convergence and Divergence:

- A sequence $\{a_n\}_{n=1}^{\infty}$is said to converge to a limit L if for any positive ε, there exists a natural number n_0 such that $|a_n - L| < ε$ for all n ≥ N.
- If a sequence does not converge, it is said to diverge.
- The limit of a sequence is unique, meaning if a sequence converges, its limit is unique.

3. Types of Convergence:

a) Convergence to a Limit:

- A sequence $\{a_n\}_{n=1}^{\infty}$converges to a limit L if, as n approaches infinity, the terms of the sequence get arbitrarily close to L.
- The notation used is $\lim_{n\to\infty} a_n = L \; or \; a_n \to L \; as \; n \to \infty$.

b) Divergence to Infinity:

- A sequence $\{a_n\}_{n=1}^{\infty}$diverges to infinity if, for any M, there exists an index N such that $a_n > M \; for \; all \; n \geq N$.
- The notation used is $\lim_{n\to\infty} a_n = \infty \; or \; a_n \to \infty \; as \; n \to \infty$.

c) Oscillation:

- A sequence $\{a_n\}_{n=1}^{\infty}$ is said to oscillate if it does not converge and does not diverge to infinity. The terms of the sequence keep alternating without approaching any specific value.

4. Properties of Convergent Sequences:

a. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

   i. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

   ii. $\lim_{n\to\infty} (c\, a_{n)} = cL$, where c is a constant

   iii. $\lim_{n\to\infty} (a_n * b_n) = L * M$

iv. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ $(if\ M \neq 0\ and\ b_n \neq 0\ for$ all n)

5. Series:

a. A series is the sum of the terms of a sequence.

b. The partial sum of a series is the sum of a finite number of terms of the sequence.

c. A series converges if the sequence of partial sums converges. Otherwise, it diverges.

6. Tests for Convergence and Divergence of Series:

a) Geometric Series:

d. A geometric series has the form $\sum_{n=1}^{\infty} ar^n$ where a is the first term and r is the common ratio.

e. The series converges if $|r| < 1$, and the sum is given by $S = \frac{a}{1-r}$.

f. The series diverges if $|r| \geq 1$.

b) Divergence Test:

g. If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

c) Comparison Test:

h. If $0 \leq b_n \leq a_n$ for all n, and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

i. If $0 \leq a_n \leq b_n$ for all n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

d) Alternating Series Test:

j. An alternating series has the form $\sum_{n=1}^{\infty}(-1)^n a_n$ where $a_n > 0$ for all n.

k. If the terms decrease $\left(i.e., a_n \geq a_{\{n+1\}}\right)\ and\ \lim_{n\to\infty} a_n = 0$, then the series converges.

e) Ratio Test:

l. If $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right) < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

m. If $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right) > 1$ or the limit does not exist, then the series diverges.

n. If $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right) = 1, the$ test is inconclusive.

Applications:

a. Sequences and series are extensively used in calculus, analysis, and various branches of mathematics to study functions, approximation, and numerical methods.

b. They are employed in areas such as physics, engineering, finance, and computer science for modelling and solving real-world problems.

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