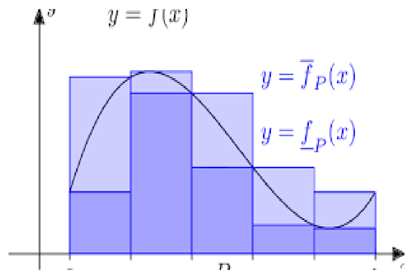
B.Sc. Mathematics (Honours)

Class Notes: Riemann Integration



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RIEMANN INTEGRATION

1. Introduction:

Riemann integration is a fundamental concept in calculus that allows us to compute the area under a curve or the integral of a function over a given interval. It was developed by the German mathematician Bernhard Riemann in the mid-19th century and has since become a cornerstone of modern analysis. Riemann integration provides a rigorous framework for defining and evaluating integrals, enabling us to study the behavior of functions and solve a wide range of mathematical problems.

We first need some basic definitions and terminology.

2. Partition of an Interval:

Let $a, b \in \mathbb{R}$ with $a < b$. A partition of the interval $[a, b]$ is a set $P = \{x_0, x_1, x_2, \dots, x_n\}$ of finitely many points of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Examples: -

1 $P = \{a, b\}$ is a partition of $[a, b]$.

2 $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ is a partition of $[0,1]$.

3 For each $n \in \mathbb{N}$, the set

$$P_n = \left\{a, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{(n-1)(b-a)}{n}, b\right\}$$

it a partition of $[a, b]$ into n equal parts. This is sometimes called the equidistant partition of $[a, b]$ in n parts.

In general,

let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$.
We call $[x_{i-1}, x_i]$ the i^{th} subinterval of the partition P. Its length $= x_i - x_{i-1}$ is denoted by δ_i or Δ_i . $\| P \| = \max\{\delta_1, \delta_2, \dots, \delta_n\}$.

Given any bounded function $f: [a, b] \to \mathbb{R}$, let us define

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

and

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},\ i = 1,2, \dots, n.$$

Since f is bounded function on $[a, b]$, we define

$$m(f) := \inf\{f(x) \mid x \in [a, b]\}$$

and
$$M(f) := \sup\{f(x) \mid x \in [a, b]\}.$$

Note that

$$m(f) \leq m_i(f) \leq M_i(f) \leq M(f) \text{ for each } i \in 1,2, \ldots, n.$$

3. Riemann Sum:

Given a partition

$$P = \{x_0, x_1, \ldots, x_n\} \text{ of}$$

$[a, b]$ and a bounded function $f: [a, b] \rightarrow \mathbb{R}$, we define the lower sum of f w.r.t P to be

$$L(P, f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

and the "upper sum" of f w.r.t P to be

$$U(P, f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

Results: -

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function.

(i) If P is a partition of $[a, b]$ and P^* is a refinement of P, i.e., $P \subseteq P^*$, then

$$L(P, f) \leqslant L(P^*, f) \;\&\; U(P, f) \geqslant U(P^*, f).$$

In short, lower sums increase and the upper sums decrease as the partition becomes finer and finer.

(ii) If P_1, P_2 are any two partitions of $[a, b]$, then $L(P_1, f) \leqslant U(P_2, f)$.

(iii) Let P be any partition of $[a, b]$. Then

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

4. Riemann Integrability:

Definition: - Let $f:[a,b] \to \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a,b]$.

The lower Riemann Integral of f on $[a,b]$ is defined as Sup $\{L(p,f) \mid P$ is a partition of $[a,b]\}$, and is denoted by $\underline{\int}_a^b f(x)dx$.

i.e.,

$$\begin{aligned} \underline{\int}_a^b f(x)dx &= \text{Sup}\{L(p,f) \mid P \text{ a partition of } [a,b]\} \\ &= \text{Sup}_P\, L(P,f) \end{aligned}$$

The upper Riemann Integral of f on $[a,b]$ is defined as Inf $\{U(p,f) \mid P$ a partition of $[a,b]\}$, and is denoted by $\overline{\int}_a^b f(x)dx$.

i.e., $\overline{\int}_a^b f(x)dx = \text{Inf}\{U(P,f) \mid P \text{ a partition } \{[\mathrm{a},\mathrm{b}]\}$

$$= \text{Inf}_P\, U(P,f)\ .$$

Definition: - (Riemann Integrable) A bounded function $f:[a,b] \to \mathbb{R}$ is said to be "Riemann Integrable" over $[a,b]$ if

$$\underline{\int}_a^b f(x)dx = \overline{\int}_a^b f(x)dx, \text{ and it is}$$

denoted by $\int_a^b f(x)dx$.

Remarks:

1 If $f:[a,b] \to \mathbb{R}$ is a bounded function, then $\underline{\int}_a^b f(x)dx \le \overline{\int}_a^b f(x)dx$.

2 A bounded function f is Riemann integrable on $[a,b] \Leftrightarrow \underline{\int}_a^b f(x)dx = \overline{\int}_a^b f(x)dx$.

3 If a bounded function f is such that $\underline{\int}_a^b f(x)dx \neq \overline{\int}_a^b f(x)dx$, then f is not Riemann integrable on $[a,b]$.

Examples: -

1. A constant function is Riemann integrable on $[\boldsymbol{a},\boldsymbol{b}]$.

Solution:- Let $f(x) = k \ \ \forall x \in [a,b]$, where $k \in \mathbb{R}$ be a constant.

Clearly f is bounded on $[a,b]$ and $\inf f = \sup f = k$.

let $P = \{x_0, x_1, \dots, x_n\}$ be a partition on $[a,b]$.

Let m_γ, M_γ be the inf and sup of f on I_γ, where $I_\gamma = [x_{\gamma-1}, x_\gamma]$.

Since $f(x) = k\ \forall x \in [a,b]$, $m_r = M_\gamma = k$.

$$\therefore L(P,f) = \sum_{\gamma=1}^{n} m_\gamma \delta_\gamma = k \sum_{\gamma=1}^{n} \delta_\gamma = k(b-a)$$

$$U(P,f) = \sum_{\gamma=1}^{n} M_\gamma \delta_\gamma = k \sum_{\gamma=1}^{n} \delta_\gamma = k(b-a).$$

$$\therefore \underline{\int}_a^b f(x)dx = \sup_p \{L(p,f)\} = k(b-a).$$

Similarly, $\overline{\int}_a^b f(x)dx = \inf_p \{\cup(p,f)\} = k(b-a)$.

$\therefore \int_a^b f(x)dx = \underline{\int}_a^b f(x)dx = \overline{\int}_a^b f(x)dx = k(b-a)$.

$\therefore f$ is Riemann integrable on $[a,b]$ and $\int_a^b k dx = k(b-a)$.

2. The function $\boldsymbol{f}(\boldsymbol{x}) = \begin{cases} \mathbf{1} & \text{when } \boldsymbol{x} \in \mathbb{R} \\ \mathbf{-1} & \text{when } \boldsymbol{x} \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$ is not Riemann integrable on $[\boldsymbol{a},\boldsymbol{b}]$.

Solution: - By the definition of the function f,

$$-1 \le f(x) \le 1\ \forall x \in [a,b].$$

$\therefore f$ is bounded on $[a, b]$ and $\inf f = -1$, $\sup f = 1$.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ and m_r, M_r be the inf and sup of f on $I_r = [x_{r-1}, x_r]$.

$$\therefore m_r = -1 \text{ and } M_\gamma = 1 \text{ for } r = 1,2, \dots, n.$$

$$\therefore L(P, f) = \sum_{r=1}^{n} m_r \delta_r = -1. \sum_{r=1}^{n} \delta_r = (-1)(b-a)$$

and

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \cdot \delta_r = (b-a).$$

$\because L(P, f) = -(b-a)$ is constant, $\int_{-a}^{b} f(x)dx = -(\mathrm{b}-\mathrm{a})$. Since $U(P, f) = (b-a)$ is constant, $\int_{a}^{\bar{b}} f(x)dx = (b-a)$.

$\therefore$ Lower and upper integrals exist but are not equal

$\Rightarrow f$ is not Riemann integrable on $[a, b]$.

Note: 1 A function need not be Riemann integrable on $[a, b]$ even though it is bounded on $[a, b]$.

2 If f is Riemann integrable on $[a, b]$ and m, M are the infimum and supremum of f on $[a, b]$, then

$$m(b-a) \leq \int_{a}^{b} f(x)dx \leq M(b-a).$$

3 If $f: [a, b] \to \mathbb{R}$ is a bounded function, then for each $\mathcal{E} > 0 \;\exists\; \delta > 0$ such that

(i) $U(P, f) < \int_{a}^{\bar{b}} f(x)dx + \varepsilon$ and

(ii) $L(P, f) > \int_{a}^{\bar{b}} f(x)dx - \varepsilon$

for each $P \in \wp[a, b]$ (set of all partitions on $[a, b]$) with $\|P\| < \delta$.

4 Let $f: [a, b] \to \mathbb{R}$ be a bounded function. As the norm of a partition P, $\| P \|$, becomes small, the number of partition points becomes large in such a way that $n \to \infty$ as $\| P \| \to 0$. Hence $\int_{a}^{\bar{b}} f(x)dx = \lim_{\|P\| \to 0} U(P, f) = \lim_{n \to \infty} U(P, f)$

A Necessary and Sufficient condition for Integrability

A bonded function $f:[a,b]\rightarrow\mathbb{R}$ is Riemann integrable on [a, b] if and only if for each $\epsilon >$ $0\ \exists\ a$ partition of P of [a, b] such that

$$0 \le U(P,f) - L(P,f) < \varepsilon.$$

Proof: -

Necessary part:

Let f be Riemann integroble on $[\mathrm{a}, b]$.

$$\therefore \underline{\int}_{\mathrm{a}}^{b} f(x)dx = \overline{\int}_{a}^{b} f(x)dx = \int_{0}^{b} f(x)dx. \qquad (1)$$

Let $\epsilon > 0$.
By Darboux's theorem $\exists\ \delta > 0$ such that

$$U(P,f) < \overline{\int}_{a}^{\mathrm{b}} f(x)dx + \frac{\epsilon}{2} \qquad (2)$$

and

$$L(P,f) > \underline{\int}_{\mathrm{a}}^{\mathrm{b}} f(x)dx - \frac{\epsilon}{2} \qquad (3)$$

for each $P \in \wp[a,b]$ with $\| P \| < \epsilon$.

From (1) & (2), we get

$$U(p,f) < \int_{a}^{b} f(x)dx + \frac{\epsilon}{2}$$

Again, from (1) & (3), we have

$$\int_{\mathrm{a}}^{\mathrm{b}} f(x)dx < L(P,f) + \frac{\epsilon}{2}$$

$$\therefore U(r,f) < \left(L(p,f) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2}$$

$$\Rightarrow U(P,f) - L(P,\mathrm{f}) < \varepsilon$$

Also, $U(P,f) - L(P,\mathrm{f}) \geqslant 0$.

$$\therefore 0 \le U(P,f) - L(P,\mathrm{f}) < \varepsilon.$$

Sufficient part:

Let for each $e > 0\ \exists\ P \in [\mathrm{a}, b]$ such that

$$0 \leq U(P,f) - L(P,f) < \varepsilon$$

By definition, $\bar{\int}_a^b f(x)dx = \inf\{U(P,f) \mid P \in \wp[[a,b]\}$

$$\Rightarrow \bar{\int}_a^b f(x)dx \leqslant U(P,f).$$

By definition.

$$\underline{\int}_a^b f(x)dx = \sup\{L(P,f) \mid P \in \wp[a,b]\}$$

$$\Rightarrow \underline{\int}_a^b f(x)dx \geq L(P,f)$$

$$\Rightarrow -\int_a^b f(x)dx \leq -L(P,f)$$

$$\therefore 0 \leq \bar{\int}_a^b f(x)dx - \underline{\int}_a^b f(x)dx \leqslant U(P,f) - L(p,f) < \varepsilon.$$

$\therefore$ For each $\epsilon > 0$, we have $0 \leq \bar{\int}_a^b f(x)dx - \underline{\int}_a^b f(x)dx < e$

$\Rightarrow \bar{\int}_a^b f(x)dx = \underline{\int}_a^b f(x)dx.$

$\therefore f$ is Riemann integrable on $[a,b]$.

5. Riemann Integrability Criteria:

a) Boundedness: A function f must be bounded on $[a,b]$ for Riemann integrability.

b) Finiteness of Discontinuities: A function f must have only finitely many discontinuities (either removable or jump) on [a, b] for Riemann integrability.

c) Finite Number of Discontinuity Jumps: If f has jump discontinuities, the sum of the sizes of all jumps must be finite for Riemann integrability.

d) Integrability of Discontinuity Points: If f has discontinuities, the function must be integrable at each of those points.

6. Properties of Riemann Integrals:

a) Linearity: $\int_a^b \{cf(x) + dg(x)\}dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$ for any constants c, d and functions $f(x)$ and $g(x)$.

b) Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

c) Monotonicity: If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable on (a, b) and $F'(x) = f(x) \forall x \in (a, b)$.

7. Conclusion:

Riemann integration provides a rigorous framework for evaluating integrals and computing areas under curves. By partitioning the interval, selecting sample points, and constructing Riemann sums, we can approximate the definite integral of a function. Riemann integrals have various properties that make them useful tools for solving mathematical problems and analyzing functions. Understanding Riemann integration is essential for advanced calculus, real analysis, and many other branches of mathematics.

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