THEORY ERROR

PART-2

1.21 ERROR DETERMINATION

We know that $\lim\limits_{\partial x \to 0} \dfrac{\partial y}{\partial x} = \dfrac{dy}{dx}$

$$\frac{\partial y}{\partial x} = \frac{dy}{dx} \text{ approximately} \Rightarrow \delta y = \left(\frac{dy}{dx}\right).\delta x \text{ approximately}$$

Definitions:

(*i*) δx is known as *absolute error* in x.

(*ii*) $\dfrac{\delta x}{x}$ is known as *relative error* in x.

(*iii*) $\left(\dfrac{\delta x}{x}\right) \times 100$ is known as *percentage error* in x.

Example 51. *The power dissipated in a resistor is given by* $P = \dfrac{E^2}{R}$. *Find by using Calculus the approximate percentage change in P when E is increased by 3% and R is decreased by 2%.* (*A.M.I.E., Summer 2001*)

Solution. Here, we have $P = \dfrac{E^2}{R} \Rightarrow \log P = 2 \log E - \log R$

On differentiating, we get

$$\frac{\delta P}{P} = \frac{2}{E}\delta E - \frac{\delta R}{R} \Rightarrow 100\frac{\delta P}{P} = 2 \times \frac{100\,\delta E}{E} - \frac{100\,\delta R}{R}$$

$$100\frac{\delta P}{P} = 2(3) - (-2) = 8 \qquad \left[\text{Given}, \frac{100\,\delta E}{E} = 2\%, \frac{100\,\delta R}{R} = -2\%\right]$$

Percentage change in $P = 8\%$ **Ans.**

Example 52. *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 and 64 cm respectively. The possible error in each measurement is ± 5%. Find approximately the maximum possible error in the computed value for the volume and the lateral surface. Find the corresponding percentage error.*

Solution. Here we have, Diameter of the can $(D) = 40$ cm

$$\frac{100\,\delta D}{D} = \frac{100\,\delta h}{h} = \pm 5\%$$

$$V = \pi r^2 h = \frac{\pi (D^2) h}{4} = \frac{\pi}{4} D^2 h$$

$$\log V = \log\frac{\pi}{4} + 2\log D + \log h$$

$$\frac{\delta V}{V} = 0 + \frac{2\,\delta D}{D} + \frac{\delta h}{h}$$

$$\frac{\delta V}{V}100 = \frac{2\delta D}{D}100 + \frac{\delta h}{h}100 = 2(\pm 5) + (\pm 5) = \pm 15$$ **Ans.**

Again

$$S = 2\pi r l = \pi D h$$

$$\log S = \log \pi + \log D + \log h$$

$$\frac{\delta S}{S} = 0 + \frac{\delta D}{D} + \frac{\delta h}{h}$$

$$\frac{\delta S}{S}100 = \frac{\delta D}{D}100 + \frac{\delta h}{h}100 = (\pm 5) + (\pm 5) = \pm 10$$ **Ans.**

Example 53. *The period T of a simple pendulum is*

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Find the maximum error in T due to possible errors upto 1% in l and 2% in g.
(*U.P. I semester winter 2003*)

Solution. We have, $T = 2\pi\sqrt{\frac{l}{g}}.$

$$\Rightarrow \quad \log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

Differentiating, we get

$$\frac{\partial T}{T} = 0 + \frac{1}{2}\frac{\partial l}{l} - \frac{1}{2}\frac{\partial g}{g}$$

$$\Rightarrow \quad \left(\frac{\partial T}{T}\right) \times 100 = \frac{1}{2}\left[\left(\frac{\partial l}{l}\right) \times 100 - \left(\frac{\partial g}{g}\right) \times 100\right]$$

$$\text{But } \frac{\partial l}{l} \times 100 = 1, \frac{\partial g}{g} \times 100 = 2$$

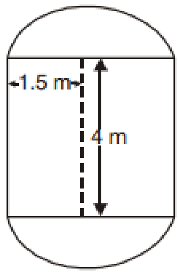
$$\therefore \quad \left(\frac{\partial T}{T}\right) \times 100 = \frac{1}{2}[1 \pm 2] = \frac{3}{2}$$

Maximum error in $T = 1.5\%$ **Ans.**

Example 54. *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of the balloon.*
(*U.P. I Sem., Dec., 2005, Comp 2002*)

Solution. Radius of the cylinder (r) = 1.5 m

Length of the cylinder (h) = 4 m



Volume of the balloon = Volume of cylinder + Volume of two hemispheres

$$\text{Volume } (V) = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = \pi\, 2r\, \delta r . h + \pi r^2 . \delta h + \frac{4}{3}\pi\, 3r^2 . \delta r$$

$$\frac{\delta V}{V} = \frac{\pi r[2\,\delta r . h + r . \delta h + 4\, r\, \delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3} = \frac{2 . \delta r . h + r . \delta h + 4r . \delta r}{r\, h + \frac{4}{3} r^2}$$

$$= \frac{2 \times 0.01 \times 4 + 1.5 \times 0.05 + 4 \times 1.5 \times 0.01}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.08 + 0.075 + 0.06}{6 + 3} = \frac{0.215}{9}$$

$$100\frac{\delta V}{V} = \frac{100 \times 0.215}{9} = \frac{21.5}{9} = 2.389\%$$ **Ans.**

Example 55. *In estimating the number of bricks in a pile which is measured to be (5m × 10m × 5m), count of bricks is taken as 100 bricks per m³. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is ₹ 2,000 per thousand bricks.* (*U.P., I Semester, Winter 2000*)

Solution. Volume $V = x\, y\, z$

$\log V = \log x + \log y + \log z$

Differentiating, we get

$$\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$100\frac{\delta V}{V} = \frac{100\,\delta x}{x} + \frac{100\,\delta y}{y} + \frac{100\,\delta z}{z} = 2 + 2 + 2$$

$$\frac{100\,\delta V}{V} = 6$$

$$\delta V = \frac{6V}{100} = \frac{6\,(5 \times 10 \times 5)}{100} = 15 \text{ cubicmetre.}$$

Number of bricks in $\delta V = 15 \times 100 = 1500$

Error in cost $= \dfrac{1500 \times 2000}{1000} = 3000$

Thus error in cost, a loss to the seller of bricks = ₹ 3000. **Ans.**

Example 56. *The angles of a triangle are calculated from the sides a, b, c. If small changes δa, δb, δc are made in the sides, show that approximately*

$$\delta A = \frac{a}{2\,\Delta}\left[\delta a - \delta b.\cos C - \delta c.\cos B\right]$$

where Δ is the area of the triangle and A, B, C are the angles opposite to a, b, c respectively. Verify that $\delta A + \delta B + \delta C = 0$ (*U.P., I Sem., Winter 2001, A.M.I.E.T.E., 2001*)

Solution. We know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2\,b\,c\cos A \qquad \ldots(1)$$

Differentiating both sides of (1), we get

$$2a\,\delta a = 2b\,\delta b + 2\,c\,\delta c - 2b\,\delta c\cos A - 2\,\delta b\,c\cos A + 2b\,c\sin A\,\delta A \qquad \text{(approx.)}$$

$$a\,\delta a = b\,\delta b + c\,\delta c - b\,\delta c\cos A - \delta b\,c\cos A + b\,c\sin A\,\delta A$$

$$\Rightarrow \quad bc\sin A\,\delta A = a\,\delta a - (b - c\cos A)\,\delta b - (c - b\cos A)\,\delta c$$

$$\Rightarrow \quad 2\,\Delta\,\delta A = a\,\delta a - (a\cos C + c\cos A - c\cos A)\,\delta b - (a\cos B + b\cos A - b\cos A)\,\delta c$$

$$\left[\begin{array}{l}\because \Delta = \frac{1}{2}bc\sin A \\ b\cos C + c\cos B = a\end{array}\right]$$

$$2\,\Delta\,\delta A = a\,\delta a - a\,\delta b\cos C - a\,\delta c\cos B$$

$$= a\,(\delta a - \delta b\cos C - \delta c\cos B)$$

$$\Rightarrow \quad \delta A = \frac{a}{2\,\Delta}\left[\delta a - \delta b.\cos C - \delta c.\cos B\right] \qquad \ldots(2)$$

Similarly,

$$\delta B = \frac{b}{2\,\Delta}\left[\delta b - \delta c.\cos A - \delta a.\cos C\right] \qquad \ldots(3)$$

$$\delta C = \frac{c}{2\,\Delta}\left[\delta c - \delta a.\cos B - \delta b.\cos A\right] \qquad \ldots(4)$$

On adding (2), (3) and (4), we get

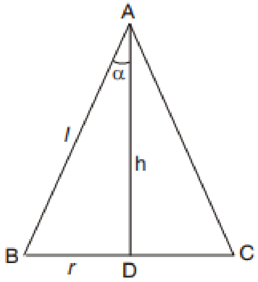
$$[\delta A + \delta B + \delta C] = \frac{1}{2\,\Delta}[(a - b\cos C - c\cos B)\,\delta a + (b - a\cos C - c\cos A)\,\delta b$$
$$+ (c - a\cos B - b\cos A)\,\delta c]$$

$$= \frac{1}{2\,\Delta}[(a - a)\,\delta a + (b - b)\,\delta b + (c - c)\,\delta c]$$

$$= 0 \qquad [\because\; b\cos C + c\cos B = a] \quad \textbf{Verified.}$$

Example 57. *The height h and semi-vertical angle α of a cone are measured, and from there A, the total area of the cone, including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that, if $\alpha = \frac{\pi}{6}$, an error of + 1 per cent in h will be approximately compensated by an error of $-19.8'$ in α.* (*A.M.I.E.T.E., Summer 2003*)

Solution. Let l be the slant height of the cone and r its radius



$$l = h \sec\alpha$$
$$r = h\tan\alpha$$
$$A = \pi r^2 + \pi r l$$
$$= \pi h^2 \tan^2\alpha + \pi (h\tan\alpha)(h\sec\alpha)$$
$$= \pi h^2 [\tan^2\alpha + \tan\alpha\sec\alpha]$$
$$\delta A = 2\pi h\,\delta h[\tan^2\alpha + \tan\alpha\sec\alpha]$$
$$+ \pi h^2[2\tan\alpha\sec^2\alpha\,\delta\alpha + \sec^2\alpha.\,\delta\alpha.\,\sec\alpha + \tan\alpha\sec\alpha\tan\alpha\,\delta\alpha]$$
$$\delta A = 2\pi h[\tan\alpha + \sec\alpha]\tan\alpha.\,\delta h + \pi h^2[2\tan\alpha\sec\alpha + \sec^2\alpha + \tan^2\alpha].\sec\alpha.\,\delta\alpha$$
$$\delta A = 2\pi h[\tan\alpha + \sec\alpha]\tan\alpha.\,\delta h + \pi h^2[\tan\alpha + \sec\alpha]^2\sec\alpha.\,\delta\alpha$$
$$\delta A = \pi h^2[\tan\alpha + \sec\alpha]\left[2\tan\alpha\frac{\delta h}{h} + (\tan\alpha + \sec\alpha)\sec\alpha.\delta\alpha\right]$$

On putting $\delta A = 0, \Rightarrow \alpha = \frac{\pi}{6}, \frac{\delta h}{h}\times 100 = 1$, we get

$$\Rightarrow \quad 0 = \pi h^2\left[\tan\frac{\pi}{6} + \sec\frac{\pi}{6}\right]\left[\left(2\tan\frac{\pi}{6}\right)\frac{1}{100} + \left(\tan\frac{\pi}{6} + \sec\frac{\pi}{6}\right)\sec\frac{\pi}{6}\,\delta\alpha\right]$$

$$\Rightarrow \quad 0 = \left[\left(2\tan\frac{\pi}{6}\right)\frac{1}{100} + \left(\tan\frac{\pi}{6} + \sec\frac{\pi}{6}\right)\sec\frac{\pi}{6}\,\delta\alpha\right]$$

$$\Rightarrow \quad 0 = \frac{2}{\sqrt{3}}\frac{1}{100} + \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}\delta\alpha \Rightarrow \frac{2}{\sqrt{3}}\frac{1}{100} - \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\frac{2}{\sqrt{3}}\delta\alpha$$

$$\Rightarrow \quad \frac{1}{100} = -\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\delta\alpha \Rightarrow \frac{1}{100} = \frac{-3}{\sqrt{3}}\delta\alpha \Rightarrow \delta\alpha = -\frac{1}{100\sqrt{3}}$$

$$\Rightarrow \quad \delta\alpha = -\frac{1}{100\sqrt{3}}\frac{180}{\pi}\text{ degree} = \left(-\frac{9}{5\sqrt{3}}\right)\frac{60}{\pi}\text{ minutes} = -19.8\text{ minutes}$$ **Ans.**

Example 58. *Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ if r_1, r_2, r_3 are each in error by plus 1.2%.*

Solution. Here, $$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \quad ...(1)$$

Differentiating, we get

$$-\frac{1}{r^2}dr = -\frac{1}{r_1^2}dr_1 - \frac{1}{r_2^2}dr_2 - \frac{1}{r_3^2}dr_3$$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{100\,dr}{r}\right) = \frac{1}{r_1}\left(\frac{100\,dr_1}{r_1}\right) + \frac{1}{r_2}\left(\frac{100\,dr_2}{r_2}\right) + \frac{1}{r_3}\left(\frac{100\,dr_3}{r_3}\right)$$

$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2) = (1.2)\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

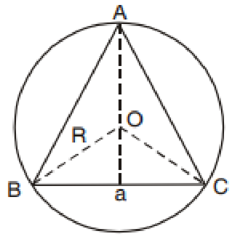
$$= 1.2\left(\frac{1}{r}\right) \quad \text{[From (1).]}$$

$$\Rightarrow \quad \frac{100\,dr}{r} = 1.2\%$$ **Ans.**

Example 59. *If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant, prove that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$, where da, db, dc are small* *increments in the sides, a, b, c respectively.*

Solution. From the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



We know that $$R = \frac{a}{2 \sin A}, \qquad \ldots(1)$$

Differentiating, we get

$$\frac{\partial R}{\partial A} = -\frac{a \cos A}{2 \sin^2 A}$$

$$\frac{\partial R}{\partial a} = \frac{1}{2 \sin A}$$

By total differentiation $$dR = \frac{\partial R}{\partial A} dA + \frac{\partial R}{\partial a} da$$

$$0 = -\frac{a \cos A}{2 \sin^2 A} . dA + \frac{1}{2 \sin A} . da, \qquad R \text{ being constant}$$

$$\Rightarrow \quad \frac{\cos A}{\sin A} dA = \frac{1}{\sin A} da$$

$$\Rightarrow \quad \frac{da}{\cos A} = \frac{a}{\sin A} . dA = 2R . dA \qquad \text{[Using (1)]}$$

$$\Rightarrow \quad \frac{da}{\cos A} = 2 R\, d A \qquad \ldots(1)$$

Similarly, $$\frac{db}{\cos B} = 2 R\, d B \qquad \ldots(2)$$

and $$\frac{dc}{\cos C} = 2 R\, d C \qquad \ldots(3)$$

Adding (1), (2) and (3), we have

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,[dA + dB + dC] \qquad \ldots(4)$$

But in any triangle ABC, $A + B + C = \pi$

Hence, $dA + dB + dC = 0$

Putting value of $dA + dB + dC = 0$ in (4), we get

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(0) = 0 \Rightarrow \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

Proved.

Example 60. *Compute an approximate value of* $(1.04)^{3.01}$.

Solution. Let $f(x, y) = x^y$

We have $$\frac{\partial f}{\partial x} = y\, x^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \log x$$

Here, let $x = 1,\ \delta x = 0.04,$

$$y = 3,\ \delta x = 0.01 \qquad \ldots(1)$$

Now $$d f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \qquad \ldots(2)$$

$$= y\, x^{y-1} + x^y \log x$$

Substituting the values from (1) in (2), we get

$$d f = (3)\,(1)^{3-1}\,(0.04) + (1)^3 \log (1)\,(0.01) = 0.12$$

$$(1.04)^{3.01} = f(1, 3) + d f = 1 + 0.12 = 1.12$$

Ans.

Example 61. *Find* $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$

Solution. Let $f(x, y) = (x^2 + 2y^3)^{\frac{1}{5}}$

Taking $x = 4,\ \delta x = 3.82 - 4 = -0.18$

$y = 2,\ \delta x = 2.1 - 2 = 0.1$

$$\frac{\partial f}{\partial x} = \frac{1}{5}[x^2 + 2y^3]^{-\frac{4}{5}}(2x) = \frac{2}{5}(4)[16 + 2(2)^3]^{-\frac{4}{5}} = \frac{8}{5}\left(\frac{1}{16}\right) = \frac{1}{10}$$

$$\frac{\partial f}{\partial y} = \frac{1}{5}[x^2 + 2y^3]^{-\frac{4}{5}}(6y^2) = \frac{6}{5}(2)^2[16 + 2(2)^3]^{-\frac{4}{5}} = \frac{24}{5} \times \frac{1}{16} = \frac{3}{10}$$

By total differentiation, we get

$$df = \frac{\partial f}{\partial x}\partial x + \frac{\partial f}{\partial y}\partial y = \frac{1}{10}(-0.18) + \frac{3}{10}(0.1) = -0.018 + 0.03 = 0.012$$

$$[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}} = f(4, 2) + df$$

$$= [(4)^2 + 2(2)^3]^{\frac{1}{5}} + 0.012 = 2 + 0.012 = 2.012$$ **Ans.**

EXERCISE 1.9

1. If the density ρ of a body be inferred from its weights W, ω in air and water respectively, show that the relative error in ρ due to errors δW, $\delta \omega$ in W, ω is $\frac{\delta\rho}{\rho} = \frac{-\omega}{W-\omega}.\frac{\delta W}{W} + \frac{\delta\omega}{W-\omega}$.

2. The period of oscillation of a pendulum is computed by the formula

$T = 2\pi\sqrt{\frac{l}{g}}$.

Show that the percentage error in $T = \frac{1}{2}$ [% error in l – % error in g]

If $l = 6$ cm and relative error in g is equal to $\frac{1}{160}$, find the error in the determination of T.

(Given $g = 981$ cm/sec^2) **Ans.** – 0.00153

3. The indicated horse power I of an engine is calculated from the formula.

$I = PLAN/33000$

where $A = \frac{\pi}{4}d^2$. Assuming that errors of r percent may have been made in measuring P. L, N and d. Find the greatest possible error in I. **Ans.** $5\,r$ %

4. The dimensions of a cone are radius 4 cm, height 6 cm. What is the error in its volume if the scale used in taking the measurement is short by 0.01 cm per cm. **Ans.** $0.96\ \pi$ cm^3.

5. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^2\ D^{2/3}\ t^2$.

Find approximately the percentage increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance is increased by 3%. **Ans.** $\frac{14}{3}$%

6. The power P required to propel a ship of length l moving with a velocity V is given by $P = kV^3\ l^2$. Find the percentage increase in power if increase in velocity is 3% and increase in length is 4%. **Ans.** 17%

7. In estimating the cost of a pile of bricks measured as 2m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length if the count is 450 bricks to 1 m^3 and bricks cost ₹ 1300 per 1000, find the approximate error in the cost. **Ans.** ₹ 631.80

8. In estimating the cost of a pile of bricks measured as 6′ × 50′ × 4′, the tape is stretched 1% beyond the standard length. If the count is 12 bricks to ft^3, and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (*U.P. I Sem., Dec. 2004*) **Ans.** 720 bricks, ₹ 25.20

9. The sides of a triangle are measured as 12 cm and 15 cm and the angle included between them as 60°. If the lengths can be measured within 1% accuracy while the angle can be measured within 2% accuracy. Find the percentage error in determining (*i*) area of the triangle (*ii*) length of opposite side of the triangle. (*A.M.I.E.T.E., Winter 2002*)

10. The voltage V across a resistor is measured with error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = \frac{V^2}{R}$ generated in the resistor is $\frac{V}{R^2}(2Rh - Vk)$. If V can be measured to an accuracy of 0.5 p.c. and R to an accuracy of 1 p.c., what is the approximate possible percentage error in W? **Ans.** Zero percent

11. Find the possible percentage error in computing the parallel resistance r of two resistance r_1 and r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, where r_1 and r_2 are both is error by + 2% each. **Ans.** 2%

12. In the manufacture of closed cylindrical boxes with specified sides a, b, c ($a \neq b \neq c$), small changes of A%, B%, C% occurred in a, b, c, respectively from box to box from the specified dimension. However, the volume and surface area of all boxes were according to specification, show that:
$$\frac{A}{a(b-c)} = \frac{B}{b(c-a)} = \frac{C}{c(a-b)}$$

13. Find the percentage error in calculating the area of ellipse $x^2/a^2 + y^2/b^2 = 1$, when error of + 1% is made in measuring the major and minor axes. **Ans.** 2%
(*U.P., I Sem, Jan 2011*)

14. If $f = x^2 y^2 z^{\frac{1}{10}}$, find the approximate value of f, when $x = 1.99$, $y = 3.01$ and $z = 0.98$.
Ans. 107.784

15. A diameter and altitude of a can in the form of right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.
(*A.M.I.E., Summer 2001*) **Ans.** 5.0336 cm^3, 3.146 cm^2

16. Prove that the relative error of a quotient does not exceed the sum of the relative errors of the dividend and the divisor. (*A.M.I.E., Winter 2001*)