RELATION OF BETA-GAMMA FUNCTION

21.7 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that

$$\Gamma l = \int_0^\infty e^{-x} x^{l-1}\, dx, \qquad \frac{\Gamma l}{z^l} = \int_0^\infty e^{-zx} x^{l-1}\, dx$$

$$\Gamma l = \int_0^\infty z^l e^{-zx} x^{l-1}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^\infty e^{-z} \,.\, z^{m-1} \,.\, z^l \,.\, e^{-zx} x^{l-1}\, dx$$

$$\Gamma l \,.\, e^{-z} \,.\, z^{m-1} = \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx$$

Integrating both sides w.r.t. 'x' we get

$$\int_0^\infty \Gamma l\, e^{-z} z^{m-1}\, dz = \int_0^\infty \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx\, dz$$

$$\Gamma l\, \Gamma m = \int_0^\infty x^{l-1}\, dx \int_0^\infty e^{-(1+x)z} z^{l+m-1}\, dz$$

$$= \int_0^\infty x^{l-1}\, dx \,.\, \frac{\Gamma(l+m)}{(1+x)^{l+m}}$$

$$\Gamma l\, \Gamma m = \Gamma(l+m) \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx = \Gamma(l+m) \,.\, \beta(l, m)$$

\

$$\beta(l, m) = \frac{\Gamma l\, \Gamma m}{\Gamma(l+m)}$$

This is the required relation.

Example 11. *Show that*

$$\int_0^{\frac{\pi}{2}} \sin^P \theta \cos^q \theta\, d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\, \Gamma\left(\frac{P+q+2}{2}\right)}$$

Solution. We know that

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \qquad \ldots(1)$$

Putting $x = \sin^2 \theta, \quad dx = 2 \sin\theta \cos\theta\, d\theta$

and $1 - x = 1 - \sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta(m, n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta\, 2 \sin\theta \cos\theta\, d\theta$$

or $$\frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Putting $2m - 1 = p, \quad i.e. \quad m = \dfrac{p+1}{2}$

and $2n - 1 = q, \quad i.e. \quad n = \dfrac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{P+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^P\theta \,.\, \cos^q\theta \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^P\theta \,.\, \cos^q\theta \, d\theta = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}}$$ **Proved**

Example 12. *Find the value of* $\Gamma\frac{1}{2}$.

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}}$$

Putting $P = q = 0$ $$\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 1}$$

or $$[\,\theta\,]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \quad \text{or} \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

or $$\left(\Gamma\frac{1}{2}\right)^2 = \pi \quad \text{or} \quad \Gamma\frac{1}{2} = \sqrt{\pi}$$ **Ans.**

Example 13. *Show that*

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cot\theta)}\, d\theta = \frac{1}{2}\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P x \cos^q x \, dx = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}} \qquad \text{...(1)}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

On applying formula (1), we have

$$= \frac{\Gamma\frac{-1/2+1}{2}\,\Gamma\frac{1/2+1}{2}}{2\,\Gamma\frac{-1/2+1/2+2}{2}} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma 1} = \frac{1}{2}\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$ **Proved**

Example 14. *Evaluate* $\int_{-1}^{+1} (1+x)^{P-1} (1-x)^{q-1}\, dx.$

Solution. Put $x = \cos 2\theta$, then $dx = -2 \sin 2\theta\, d\theta$

$$\int_{-1}^{+1} (1+x)^{P-1} (1-x)^{q-1}\, dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{P-1} (1-\cos 2\theta)^{q-1} (-2 \sin 2\theta\, d\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} (1+2\cos^2\theta - 1)^{P-1} (1-1+2\sin^2\theta)^{q-1} (-4 \sin\theta \cos\theta\, d\theta)$$

$$= 4\int_{0}^{\frac{\pi}{2}} 2^{P-1} \cos^{2P-2}\theta \,.\, 2^{q-1} \sin^{2q-2}\theta \,.\, \sin\theta\cos\theta\, d\theta$$

$$= 2^{P+q} \int_{0}^{\pi} \sin^{2q-1}\theta \cos^{2P-1}\theta\, d\theta$$

$$= 2^{P+q} \frac{\Gamma\left(\frac{2q}{2}\right)\Gamma\left(\frac{2P}{2}\right)}{2\,\Gamma\left(\frac{2P+2q}{2}\right)} = 2^{P+q-1} \frac{\Gamma P\,\Gamma q}{\Gamma(P+q)}$$ **Ans.**

Example 15. *Show that* $\Gamma n\, \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$ $\quad (0 < n < 1)$

Solution. We know that

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting $m + n = 1$ or $m = 1 - n$

$$\frac{\Gamma(1-n)\, \Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)^1}\, dx$$

$$\Gamma(1-n)\, \Gamma n = \int_0^\infty \frac{x^{n-1}}{1+x}\, dx \qquad \left[\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sin n\pi}$$ **Proved**

Example 16. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}.$

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $\quad dx = \frac{2}{n} \sin^{2/n - 1}\theta \cos\theta\, d\theta$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta}{(1-\sin^2\theta)^{1/n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta \cos\theta\, d\theta}{(\cos^2\theta)^{1/n}}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta \cos^{1-2/n}\theta\, d\theta$$

$$= \frac{2}{n}\,\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right)}$$

$$= \frac{1}{n}\,\frac{\Gamma\frac{1}{n}\,\Gamma\frac{n-1}{n}}{\Gamma 1} \qquad \left(\Gamma\frac{1}{n}\,\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right)$$

$$= \frac{\pi}{n\sin\frac{\pi}{n}}$$

Ans.

Example 17. Show that $\int_0^{\frac{\pi}{2}} tan^P\,\theta\, d\,\theta = \frac{\pi}{2}\, sec\frac{P\,\pi}{2}$ *and indicate the restriction on the values of P.*

Solution. $\int_0^{\frac{\pi}{2}} \tan^P\theta\, d\,\theta = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^{-P}\theta\, d\,\theta$

$$= \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{-P+1}{2}}{2\,\Gamma\frac{P+1-P+1}{2}} \qquad \left[\begin{array}{r}1-P>0\\ 1>P\end{array}\right]$$

$$= \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{-p+1}{2}}{2\,\Gamma 1} \qquad \left[\begin{array}{r}1+P>0\\ P>-1\end{array}\right]$$

$$= \frac{1}{2}\,\Gamma\frac{1+p}{2}\,\Gamma\frac{-p+1}{2} \qquad \therefore\ 1>P>-1$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\frac{p+1}{2}\pi} = \frac{1}{2}\,\frac{\pi}{\cos\frac{p\,\pi}{2}} = \frac{\pi}{2}\sec\frac{p\,\pi}{2}$$

Proved

Example 18. *Prove Duplication Formula*

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m.$$

Hence show that $\beta(m, m) = 2^{1-2m}\,\beta\left(m, \frac{1}{2}\right)$ (*U.P., II Semeter, Summer 2001*)

Solution. We know that

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^q\theta\, d\,\theta$$

Putting $q = p$ we get

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{p+1}{2}}{2\,\Gamma(p+1)} = \int_0^{\frac{\pi}{2}} \sin^P\theta\cos^P\theta\, d\,\theta = \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^P\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^P}(2\sin\theta\cos\theta)^P\, d\theta = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^P\, d\theta$$

Putting $2\theta = t$, we have

$$= \frac{1}{2^P}\int_0^{\pi} \sin^P t\, \frac{dt}{2}$$

$$= \frac{1}{2^P}\cdot\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \sin^P t\, dt = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} \sin^P t\cos^0 t\, dt$$

$$= \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

or $$\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{P+1}{2}\right)}{2\,\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{P+2}{2}\right)}$$

Take $\frac{P+1}{2} = m$ or $P = 2m - 1$

or $$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \qquad ...(1)$$

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$ **Proved**

Multiplying both sides of (1) by Γm, we have

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(2m)} = 2^{1-2m}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$$ **Proved**

$$\beta(m, m) = 2^{1-2m}\,\beta\left(m, \frac{1}{2}\right)$$

Example 19. *Evaluate* $\iint_A \frac{dx\,dy}{\sqrt{xy}}$, *using the substitutions*

$$x = \frac{u}{1+v^2}, \quad y = \frac{uv}{1+v^2}$$

where A is bounded by $x^2 + y^2 - x = 0,\ y = 0,\ y > 0.$

Solution. Here $\sqrt{xy} = \sqrt{\left(\frac{u}{1+v^2}\right)\left(\frac{uv}{(1+v^2)}\right)} = \frac{u\sqrt{v}}{1+v^2}$

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

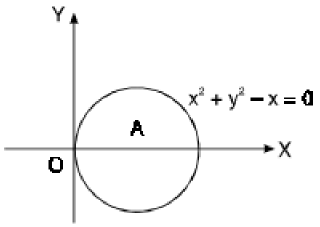
$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} \frac{1}{1+v^2} & -\frac{2uv}{(1+v^2)^2} \\ \frac{v}{1+v^2} & \frac{u(1-v^2)}{(1+v^2)^2} \end{vmatrix} du\,dv$$

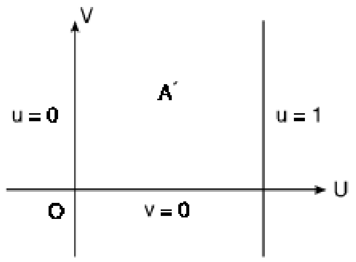
$$= \left[\frac{u(1-v^2)}{(1+v^2)^3} + \frac{2uv^2}{(1+v^2)^3}\right] du\,dv = \left[\frac{u - uv^2 + 2uv^2}{(1+v^2)^3}\right] du\,dv$$

$$= \frac{u(1+v^2)}{(1+v^2)^3}\,du\,dv = \frac{u}{(1+v^2)^2}\,du\,dv$$

Also the circle $x^2 + y^2 - x = 0$ is transformed into

$$\frac{u^2}{(1+v^2)^2} + \frac{u^2 v^2}{(1+v^2)^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad \frac{u^2(1+v^2)}{(1+v^2)^2} - \frac{u}{1+v^2} = 0$$





$$\frac{u^2}{1+v^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad u^2 - u = 0 \quad \text{or} \quad u(u-1) = 0 \Rightarrow u = 0,\ u = 1$$

Further $\quad y = 0 \Rightarrow \frac{uv}{1+v^2} = 0 \Rightarrow u = 0,\ v = 0$

and $y > 0 \Rightarrow uv > 0$ either both u and v are positive or both negative.

The area A, *i.e.*, $x^2 + y^2 - x = 0$ is transformed into A' bounded by $u = 0$, $v = 0$ and $u = 1$ *and* $v = \infty$.

$$\iint \frac{dx\,dy}{\sqrt{x}} = \int_0^1\int_0^\infty \frac{\frac{u}{(1+v^2)^2}\,du\,dv}{\sqrt{\frac{u\sqrt{v}}{1+v^2}}} = \int_0^1\int_0^\infty \frac{1}{\sqrt{v}\,(1+v^2)}\,dv\,du$$

On putting $v = \tan\theta$, $dv = sec^2\theta\,d\theta$

$$= \int_0^1\int_0^{\frac{\pi}{2}} \frac{sec^2\theta\,d\theta\,du}{\sqrt{\tan\theta}\,(1+\tan^2\theta)} = \int_0^1 du\int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}}\,d\theta = \int_0^1 du\int_0^{\frac{\pi}{2}} \sin\theta^{-\frac{1}{2}}\cos\theta^{\frac{1}{2}}\,d\theta$$

$$= \int_0^1 du\,\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma 2}\,\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma 2} = \frac{1}{2}\int_0^1 du\,\Gamma\tfrac{1}{4}\,\Gamma\tfrac{3}{4} = \frac{1}{2}\int_0^1 du\left[\frac{\sqrt{\pi}}{2^{\frac{-1}{2}}}\,\Gamma\tfrac{1}{2}\right]$$

$$= \frac{1}{2}\int_0^1 du\,\sqrt{2}\,\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi}{\sqrt{2}}\,[\,u\,]_0^1 = \frac{\pi}{\sqrt{2}}$$ **Ans.**

Example 20. *Prove that*

$$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma(l+m+1)}\, h^{l+m}$$

where D is the domain $x \geq 0,\ \ y \geq 0$ *and* $x + y \leq h.$

Solution. Putting $x = Xh$ and $y = Yh$, $dx\, dy = h^2\, dX\, dY$

$$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \iint_{D'} (Xh)^{l-1} (Yh)^{m-1} h^2\, dX\, dY$$

where D' is the domain

$X \geq 0,\ \ Y \geq 0,\ \ X + Y \leq 1$

$$= h^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1}\, dX\, dY = h^{l+m} \int_0^1 X^{l-1}\, dX \int_0^{1-X} Y^{m-1}\, dY$$

$$= h^{l+m} \int_0^1 X^{l-1}\, dX \left[\frac{Y^m}{m}\right]_0^{1-X} = \frac{h^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m}\, \beta(l,\ m+1) = \frac{h^{l+m}}{m}\, \frac{\Gamma l\, \Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= \frac{h^{l+m}}{m}\, \frac{m\, \Gamma l\, \Gamma m}{\Gamma(l+m+1)} = h^{l+m}\, \frac{\Gamma l\, \Gamma m}{\Gamma(l+m+1)}.$$ **Proved.**

Example 21. *Establish* ***Dirichlet's integral***

$$\iiint_V x^{l-1} y^{m-l} z^{n-l}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$$

where V is the region $x \geq 0,\ y \geq 0,\ z \geq 0$ *and* $x + y + z \leq 1.$

Solution. Putting $y + z \leq 1 - x = h.$ Then $z \leq h - y$

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \int_0^1 x^{l-1}\, dx \int_0^{1-x} y^{m-1}\, dy \int_0^{1-x-y} z^{n-1}\, dz$$

$$= \int_0^1 x^{l-1}\, dx \left[\int_0^h \int_0^{h-y} y^{m-1} z^{n-1}\, dy\, dz\right]$$

$$= \int_0^1 x^{l-1}\, dx \left[\frac{\Gamma m\, \Gamma n}{\Gamma(m+n+1)}\, h^{m+n}\right]$$

$$= \frac{\Gamma m\, \Gamma n}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx$$

$$= \frac{\Gamma m\, \Gamma n}{\Gamma(m+n+1)}\, \beta(l,\ m+n+1)$$

$$= \frac{\Gamma m\, \Gamma n}{\Gamma(m+n+1)}\, \frac{\Gamma l\, \Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$$ **Proved.**

Note. $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}\, h^{l+m+n}$

where V is the domain, $x \geq 0,\ y \geq 0,\ z \geq 0$ and $x + y + z \leq h$.

Exercise 21.2

Prove that

1. (*a*) $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta \, d\theta = \frac{\pi}{32}$ (*b*) $\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5\pi}{32}$

2. (*a*) $\beta(m+1, n) = \frac{m}{m+n}\beta(m, n)$ (*b*) $\beta(m, n+1) = \frac{n}{m+n}\beta(m, n)$

(*c*) $\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$

3. $\int_0^1 \sqrt{x}\,\sqrt[3]{1-x^2}\, dx = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{2\,\Gamma\left(\frac{7}{12}\right)}$

4. $\int_0^1 (1-x^n)^{-\frac{1}{2}}\, dx = \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{n\,\Gamma\left(\frac{n+2}{2n}\right)}$

5. $\int_0^1 (1-x^{1/n})^m\, dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

6. $\int_1^{\infty} \frac{dx}{x^{p+1}(x-1)^q} = \beta(p+q, 1-q)$ if $-p < q < 1$

7. $\int_0^1 x^m (1-x^n)^p\, dx = \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$

8. $\int_0^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1} \,.\, \beta(m+1, n+1)$

9. $\int_3^7 \sqrt[4]{(x-3)(7-x)}\, dx = \frac{2\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{3\sqrt{\pi}}$ Put $x = 4t+3$

10. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{4\sqrt{\pi}}$
