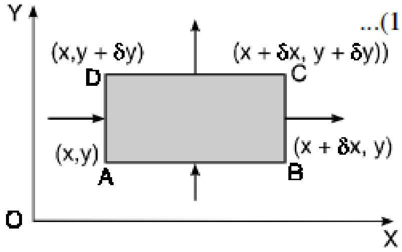
9.20 TWO DIMENSIONAL HEAT FLOW

Consider the heat flow in a metal plate of uniform thickness, in the directions parallel to length and breadth of the plate. There is no heat flow along the normal to the plane of the rectangle.

Let $u\ (x,\ y)$ be the temperature at any point $(x,\ y)$ of the plate at time t is given by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad ...(1)$$



In the steady state, u does not change with t.

$$\therefore \qquad \frac{\partial u}{\partial t} = 0$$

(1) becomes $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

This is called Laplace equation in two dimensions.

Example 13. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *which satisfies the conditions*

$$u(0,\ y) = u\ (l,\ y) = u\ (x,\ 0) = 0$$

and $\qquad u\ (x,\ a) = \sin . \dfrac{n \pi x}{l}$ *(A.M.I.E.T.E., Winter 2000)*

Solution. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$

Let $\qquad u = X(x) \,.\, Y(y) \qquad ...(2)$

Putting the values of $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$ in (1), we have

$$X''Y + XY'' = 0$$

or $\qquad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -p^2 \qquad$ (say)

$\therefore \qquad X'' = -p^2 X \qquad$ or $\qquad X'' + p^2 X = 0 \qquad ...(3)$

and $\qquad Y'' = p^2 Y \qquad$ or $\qquad Y'' - p^2 Y = 0 \qquad ...(4)$

A.E. of (3) is $\ m^2 + p^2 = 0\ $ or $\ m = \pm\ ip$

$$\therefore \qquad X = c_1 \cos p x + c_2 \sin px$$

A.E. of (4) is $\ m^2 - p^2 = 0$ or $m = \pm\ p$

$$\therefore \qquad Y = c_3\, e^{py} + c_4\, e^{-py}$$

Putting the values of X and Y in (2) we have

$$u = (c_1 \cos px + c_2 \sin px)\,(c_3\, e^{py} + c_4\, e^{-py}) \qquad ...(5)$$

Putting $x = 0$, $u = 0$ in (5) we have

$$0 = c_1\,(c_3\, e^{py} + c_4\, e^{-py})$$

$$\therefore \qquad c_1 = 0$$

(5) is reduced to $u = c_2 \sin px\,(c_3\, e^{py} + c_4\, e^{-py}) \qquad ...(6)$

On putting $x = l$, $u = 0$, we have

$$0 = c_2 \sin p\, l\,(c_3\, e^{py} + c_4\, e^{-py})$$

$$c_2 \neq 0 \qquad \therefore\ \sin p\, l = 0 = \sin n\, \pi$$

or $\qquad pl = n\,\pi \qquad$ or $\qquad p = \dfrac{n\,\pi}{l}$

Now (6) becomes

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{\frac{-n\pi y}{l}}\right) \quad ...(7)$$

On putting $u = 0$ and $y = 0$ in (7) we have

$$0 = c_2 \sin \frac{n\pi x}{l} (c_3 + c_4)$$

$\therefore \quad c_3 + c_4 = 0 \quad$ or $\quad c_3 = -c_4$

(7) becomes $u = c_2 c_3 \sin \frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}\right) \quad ...(8)$

On putting $y = a$ and $u = \sin \frac{n\pi x}{l}$ in (8), we get

$$\sin \frac{n\pi x}{l} = c_2 c_3 \sin \frac{n\pi x}{l}\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right) \qquad i.e. \quad c_2 c_3 = \frac{1}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}}$$

Putting this value in (8) we have

$$u = \sin \frac{n\pi x}{l} \frac{e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}} \quad \text{or} \quad u = \sin \frac{n\pi x}{l} \frac{\sinh \frac{n\pi y}{l}}{\sinh \frac{n\pi a}{l}} \qquad \textbf{Ans.}$$

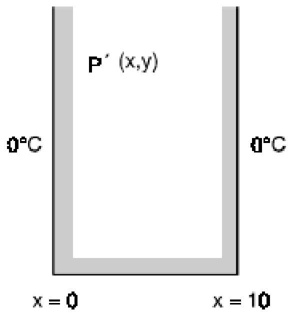
Example 14. *A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by*

$$u(x, 0) = 20x, \quad 0 < x \le 5$$
$$= 20(10 - x), \quad 5 < x < 10$$

while the two long edges $x = 0$ and $x = 10$ as well as the other short edges are kept at 0^oC. Find the steady state temperature at any point (x, y) of the plate.

Solution. In the steady state, the temperature $u(x, y)$ at any point $p(x, y)$ satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$



The boundary conditions are

$u(0, y) = 0$ for all values of y ...(2)

$u(10, y) = 0$ for all values of y ...(3)

$u(x, \infty) = 0$ for all values of x ...(4)

$u(x, 0) = 20x \quad 0 < x \le 5$

$= 20(10 - x) \quad 5 < x < 10$ ...(5)

Now three possible solutions of (1) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad ...(6)$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \quad ...(7)$$

$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \quad ...(8)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (6) and (8) cannot satisfy the condition (2), (3) and (4). Thus, only possible solution is (7) *i.e.*, of the form.

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(9)$$

By (2) $\quad u(0, y) = C_1 (C_3 e^{py} + C_4 e^{-py}) = 0$ for all values of y

$\therefore \quad C_1 = 0$

$\therefore \quad$ (9) reduces to $u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py}) \quad ...(10)$

By (3) $u(10, y) = C_2 \sin 10p \, (C_3 e^{py} + C_4 e^{-py}) = 0 \qquad C_2 \neq 0$

$\therefore \quad \sin 10p = 0 \qquad \therefore \quad 10p = n\pi \quad \text{or} \quad p = \frac{n\pi}{10}$

Also to satisfy the condition (4) *i.e.*, $u = 0$ as $y \to \infty$

$$C_3 = 0$$

Hence (10) takes the form $u(x, y) = C_2 C_4 \sin px \, e^{-py}$

or $\quad u(x, y) = b_n \sin px . e^{-py} \qquad$ where $b_n = C_2 C_4$

$\therefore$ The most general solution that satisfies (2), (3) & (4) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin px \, e^{-py} \qquad \ldots(5)$$

Putting $y = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin px \qquad$ where $p = \frac{n\pi}{10}$

This requires the expansion of u in Fourier series in the interval $x = 0$ and $x = 5$ and from $x = 5$ to $x = 10$.

$$b_n = \frac{2}{10}\int_0^5 20\, x \sin px \, dx + \frac{2}{10}\int_5^{10} 20\,(10 - x) \sin px \, dx$$

$$b_n = 4\int_0^5 x \sin px \, dx + 4\int_5^{10} (10 - x) \sin px \, dx$$

$$= 4\left[x\left(\frac{-\cos px}{p}\right) - (1)\left(\frac{-\sin px}{p^2}\right)\right]_0^5 + 4\left[(10 - x)\left(\frac{-\cos px}{p}\right) - (-1)\left(\frac{-\sin px}{p^2}\right)\right]_5^{10}$$

$$= 4\left[\frac{-5\cos 5P}{P} + \frac{\sin 5P}{P^2}\right] + 4\left[0 - \frac{\sin 10P}{P^2} + \frac{5\cos 5P}{P} + \frac{\sin 5P}{P^2}\right]$$

$$= 4\left[\frac{2\sin 5P}{P^2} - \frac{\sin 10P}{P^2}\right] \qquad \left(P = \frac{n\pi}{10}\right)$$

$$= 4\left[\frac{2\sin 5 . \frac{n\pi}{10}}{\frac{n^2\pi^2}{100}} - \frac{\sin 10 . \frac{n\pi}{10}}{\frac{n^2\pi^2}{100}}\right] = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{400}{n^2\pi^2}\sin n\pi$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} = 0 \text{ if } n \text{ is even.} = \pm\frac{800}{n^2\pi^2} \text{ if } n \text{ is odd. or } b_n = \frac{(-1)^{n+1}800}{(2n-1)^2\pi^2}$$

On putting the value of b_n in (5) the temperature at any point (x, y) is given by

$$u(x, y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}\, e^{-\frac{(2n-1)\pi y}{10}} \qquad \textbf{Ans.}$$

Exercise 9.18

1. Find by the method of separation of variables, a particular solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

that tends to zero as x tends to infinity and is equal to $2\cos y$ when $x = 0$. **Ans.** $u = 2e^{-x}\cos y$

2. Solve the equation : $\quad u_{xx} + u_{yy} = 0$

$u(0, y) = u(\pi, y) = 0$ for all y,

$u(x, 0) = k \qquad 0 < x < \pi$

$$\lim_{y \to \infty} u(x, y) = 0 \qquad 0 < x < \pi$$ *(A.M.I.E.T.E., Summer 2003)*

Ans. $u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-ny},\ k = \sum_{n=1}^{\infty} b_n \sin nx$

3. Find the solution of Laplace's equation $\nabla^2 \psi = 0$ in cartesian coordinates in the region $0 \le x \le a,\ 0 \le y \le b_0$ to satisfying the conditions $y = 0$ on $x = 0,\ x = a,\ y = 0\ y = b$ and $\psi = x(a - x),\ 0 < x < a.$ *(A.M.I.E.T.E., Winter 2001)*

Ans. $\psi = \dfrac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \dfrac{1}{(2n+1)^3} \dfrac{\sin \frac{\pi x}{a} \sinh \frac{(2n+1)\pi y}{a}}{\sinh \frac{(2n+1)\pi b}{a}}$

4. An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

(A.M.I.E.T.E., Dec. 2005) **Ans.** $u(x, y) = \dfrac{4u_o}{\pi}\left[e^{-y} \sin x + \dfrac{1}{3} e^{-3y} \sin 3x + \dfrac{1}{5} e^{-5y} \sin 5x + \ldots\right]$

5. Solve $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0,$ given that

(*i*) $V = 0$ when $x = 0$ and $x = c$; (*ii*) $V \to 0$ as $y \to \infty$; (*iii*) $V = V_0$ when $y = 0$.

Ans. $V(x, y) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{c}\, e^{-\frac{n\pi y}{c}},\ V_0 = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{c}$

6. The steady state temperature distribution in a thin plate bounded by the lines $x = 0,\ x = a,\ y = 0$ and $y = \infty$, is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Obtain the steady state temperature distribution under the conditions

$$\begin{aligned} &u(0, y) = 0, && u(a, y) = 0, && u(x, \infty) = 0 \\ &u(x, 0) = x, && 0 \le x \le a/2 \\ &\quad\ \ = a - x && a/2 \le x \le a \end{aligned}$$

7. The points of trisection of a tightly stretched string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.