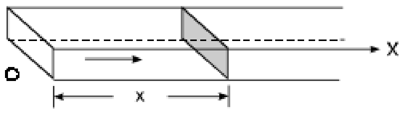
PARTIAL DIFFERENTIAL EQUATION

PART-3

9.19 ONE DIMENSIONAL HEAT FLOW



Let heat flow along a bar of uniform cross-section, in the direction perpendicular to the cross-section. Take one end of the bar as origin and the direction of heat flow is along x-axis.

Let the temperature of the bar at any time t at a point x distance from the origin be $u(x, t)$. Then the equation of one dimensional heat flow is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Example 10. *A rod of length l with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to $0^\circ C$ and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by*

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2\pi^2 n^2 t}{l^2}}$$

where b_n is determined from the equation.

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Solution. Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let us assume that $u = XT$, where X is a function of x alone and T that of t alone.

$$\therefore \quad \frac{\partial u}{\partial t} = X \frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = T \frac{d^2X}{dx^2}$$

Substituting these values in (1), we get $X \frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$

$$i.e. \quad \frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \qquad ...(2)$$

Let each side be equal to a constant $(-p^2)$.

$$\frac{1}{c^2T}\frac{dT}{dt} = -p^2 \quad \text{or} \quad \frac{dT}{dt} + p^2c^2T = 0 \qquad ...(3)$$

$$\text{and} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \quad \text{or} \quad \frac{d^2X}{dx^2} + p^2X = 0 \qquad ...(4)$$

Solving (3) and (4) we have

$$T = c_1 e^{-p^2c^2t} \quad \text{and} \quad X = c_2 \cos px + c_3 \sin px$$

$$\therefore \quad u = c_1 e^{-p^2c^2t}(c_2 \cos px + c_3 \sin px) \qquad ...(5)$$

Putting $x = 0$, $u = 0$ in (5), we get

$$0 = c_1 e^{-p^2c^2t}(c_2) \Rightarrow \quad c_2 = 0 \text{ since } c_1 \neq 0$$

(5) becomes $\quad u = c_1 e^{-p^2c^2t} c_3 \sin px \qquad ...(6)$

Again putting $x = l$, $u = 0$ in (6), we get

$$0 = c_1 e^{-p^2c^2t} \cdot c_3 \sin pl \quad \Rightarrow \sin pl = 0 = \sin n\pi$$

$$\Rightarrow pl = n\pi \quad \text{or} \quad \therefore p = \frac{n\pi}{l}, n \text{ is any integer}$$

Hence (6) becomes $u = c_1 c_3 e^{\frac{-n^2 c^2 \pi^2}{l^2} t} \sin \frac{n \pi x}{l} = b_n e^{-\frac{n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi}{l} x, \quad b_n = c_1 c_3$

This equation satisfies the given conditions for all integral values of n. Hence taking $n = 1, 2, 3, ...$, *the most general solution is*

$$u = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \sin \frac{n\pi}{l} x$$

By initial conditions $u = U_0$ when $t = 0$

$$U_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l}$$ **Proved**

Example 11. *Find the solution of* $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$

for which $u(0, t) = u(l, t) = 0$, $u(x, 0) = \sin \frac{\pi x}{l}$ *by method of variables separable.*

Solution. $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ ...(1)

In example 10 the given equation was

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad ...(2)$$

On comparing (1) and (2) we get $h^2 = \frac{1}{c^2}$

Thus solution of (1) is

$$u = (c_2 \cos px + c_3 \sin px)\, c_1 e^{-\frac{p^2 t}{h^2}} \quad \text{[Using (5) of example (10)]} \quad ...(3)$$

On putting $x = 0$, $u = 0$ in (3) we get

$$0 = c_1 c_2 \; e^{-\frac{p^2 t^2}{h^2}} \qquad c_1 \neq 0, \therefore c_2 = 0$$

(3) is reduced to

$$u = c_3 \sin px \, c_1 e^{-\frac{p^2 t}{h^2}} \quad ...(4)$$

On putting $x = l$ and $u = 0$ in (4), we get

$$0 = c_3 \sin pl \, c_1 e^{-\frac{p^2 t}{h^2}}$$

$$c_3 \neq 0, \; c_1 \neq 0 \qquad [\therefore \; \sin pl = 0 = \sin n\pi \text{ or } p = \frac{n\pi}{l}]$$

Now (4) is reduced to

$$u = c_1 \, c_3 \sin \frac{n \pi x}{l} e^{-\frac{n^2 \pi^2 t}{h^2 l^2}} \quad ...(5)$$

On putting $t = 0$, $u = \sin \frac{\pi x}{l}$ in (5) we get

$$\sin \frac{\pi x}{l} = c_4 \sin \frac{n \pi x}{l} \qquad [\text{put } c_1 \; c_3 = c_4]$$

This equation will be satisfied if

$$n = 1 \text{ and } c_4 = 1$$

On putting the values of c_4 and n in (5), we have

$$u = \sin \frac{\pi x}{l} \; e^{-\frac{\pi^2 t}{h^2 l^2}}$$ **Ans.**

Example 12. *The ends A and B of a rod 20 cm long have the temperatures at 30° C and at 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. The initial temperature distribution in the rod is

$$u = 30 + \frac{50}{20} x \qquad \text{i.e., } u = 30 + \frac{5}{2} x$$

and the final distribution (*i.e.* in steady state) is

$$u = 40 + \frac{20}{20} x = 40 + x$$

To get u in the intermediate period, reckoning time from the instant when the end temperature were changed, we assumed

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod (*i.e.* temperature after a sufficiently long time) and $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

Thus $u_2(x) = 40 + x$

Now $u_1(x, t)$ satisfies the one-dimensional heat-flow equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Hence u is of the form

$$u = 40 + x + \Sigma (a_k \cos kx + b_k \sin kx) e^{-c^2k^2t}$$

Since $u = 40°$ when $x = 0$ and $u = 60°$ when $x = 20$, we get

$$a_k = 0, \; k = \frac{n\pi}{20}$$

Hence

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-c^2\left(\frac{n\pi}{20}\right)^2 t} \qquad ...(1)$$

Using the initial condition *i.e.*,

$$u = 30 + \frac{5}{2} x \quad \text{when } t = 0, \text{ we get}$$

$$30 + \frac{5}{2} x = 40 + x + \Sigma b_n \sin \frac{n\pi}{20} x \quad \text{or} \quad \frac{3}{2} x - 10 = \Sigma b_n \sin \frac{n\pi x}{20}$$

Hence

$$b_n = \frac{2}{20} \int_0^{20} \left(\frac{3}{2} x - 10\right) \sin \frac{n\pi x}{20} dx$$

$$= \frac{1}{10} \left[\left(\frac{3x}{2} - 10\right) \left(-\frac{20}{n\pi} \cos \frac{n\pi x}{20}\right) - \frac{3}{2} \left(\frac{-400}{n^2\pi^2} \sin \frac{n\pi x}{20}\right) \right]_0^{20}$$

$$= \frac{1}{10} \left[-20 \left(\frac{20}{n\pi}\right)(-1)^n - (-10)\left(\frac{20}{n\pi}\right) \right] = -\frac{20}{n\pi} [2(-1)^n + 1]$$

Putting this value of b_n in (1) we get

$$u = 40 + x - \frac{20}{\pi} \sum \left[\left(\frac{2(-1)^n}{n}\right) \sin \frac{n\pi x}{20} \cdot e^{-\left(\frac{cn\pi}{20}\right)^2 t} \right] \qquad \textbf{Ans.}$$

EXERCISE 9.17

1. Solve the following boundary value problem which arises in the heat conduction in a rod :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0, t) = u(l, t) = 0 \qquad \text{(A.M.I.E.T.E., Summer 2002)}$$

$$u(x, 0) = 100 \frac{x}{l} \qquad \textbf{Ans. } u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2} t}$$

2. Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

subject to the boundary conditions $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$, l being the length of the bar.

Ans. $y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2c^2 t}{l^2}}$

3. Solve the non-homogeneous heat conduction equation $u = a^2 u_{xx} + \sin 3\pi x$ subject to the following conditions :

$$u(x, 0) = \sin 2\pi x, \; u(0, t) = u(l, t) = 0.$$

4. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, given that (*i*) $u = 0$ when $x = 0$ and $x = l$ for all t

(*ii*) $u = 3 \sin \frac{\pi x}{l}$, when $t = 0$ for all x, $0 < x < l$.

Ans. $u = 3 \sin \frac{\pi x}{l} \, e^{\frac{-a^2\pi^2 t}{l^2}}$

5. (*a*) Find by the method of separation of variables the solution of $U(x, t)$ of the boundary value problem

$$\frac{\partial U}{\partial t} = 3 \frac{\partial^2 U}{\partial x^2}, \quad t > 0, 0 < \alpha < 2$$
$$U(0, t) = 0, \quad U(2, t) = 0$$
$$U(x, 0) = x, \quad 0 < x < 2.$$

Ans. $U = \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{2}}{\sin \frac{n\pi}{2}} \, e^{-\frac{3n^2\pi^2 t}{4}}$

(*b*) The ends A and B of a rod 30 cm long have their temperatures kept at 20°C and 80°C respectively until steady-state conditions prevail. The temperature of the end B is suddenly reduced to 60°*C* and kept so while at the end A temperature is raised to 40°*C*. Find temperature distribution in the rod at time *t*. (*A.M.I.E.T.E., Winter 2002*)

6. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar, given that $u = u_0 \sin nt$ when $x = 0$, for all values of t and $u = 0$ when x is very large.

Without radiation, show that if $u = Ae^{-gx} \sin (nt - gx)$, where A, g and n are positive constants,

then $g = \sqrt{\frac{n}{2\mu}}$

7. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state condition prevails. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t, solve the equation of heat

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

by the method of separation of variables and obtain the solution (*A.M.I.E., Summer 2004*)

Ans. $u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2n^2\pi^2}{l^2} \cdot t}$

8. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions

$$u'(0, t) = 0 \quad t > 0$$
$$u'(\pi, t) = 0$$
$$u(x, 0) = x^2, \; 0 < x < \pi$$

Ans. $u(x, t) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \, e^{-a^2 n^2 t}$

9. A rod of length l has its lateral surface insulated and is so thin that heat flow in the rod can be regarded as one dimensional. Initially the rod is at the temperature 100 throughout. At $t = 0$ the temperaure at the left end of the rod is suddenly reduced to 50 and maintained thereafter at this value, while the right end is maintained at 100. Let $u(x, t)$ be the temperature at point x in the rod at any subsequent time t.

(*i*) Write down the appropriate partial differential equation for $u(x, t)$ with initial and boundary conditions.

(*ii*) Solve the differential equation in (*i*) above using method of separation of variables and show that

$$u(x, t) = 50 + \frac{50x}{l} + \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} \exp \frac{-n^2\pi^2 t}{a^2 l^2}$$

Where a^2 is the constant involved in the partial differential equation. (*A.M.I.E.T.E., Dec. 2004*)