# PARTIAL DIFFERENTIAL EQUATION

PART-1

## Partial Differential Equations in Practical Problems

### 9.15 INTRODUCTION

In practical problems, the following types of equations are generally used :

(*i*) *Wave equation* : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(*ii*) *One-dimensional heat flow* : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(*iii*) *Two-dimensional heat flow* : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(*iv*) *Radio equations* : $-\frac{\partial V}{\partial x} = L \frac{\partial I}{\partial t}, \quad -\frac{\partial I}{\partial x} = C \frac{\partial V}{\partial t}$

### 9.16 METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

Example 1. *Using the method of separation of variables, solve*

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$$

where $u(x, 0) = 6 e^{-3x}$ *(A.M.I.E.T.E., Winter 1999)*

Solution. $$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \qquad \text{...(1)}$$

Let $$u = X(x).T(t) \qquad \text{...(2)}$$

where $X$ is a function of $x$ only and $T$ is a function of $t$ only.

Putting the value of $u$ in (1) we get

$$\frac{\partial (X.T)}{\partial x} = 2 \frac{\partial}{\partial t}(X.T) + X.T$$

$$T \frac{dX}{dx} = 2X \frac{dT}{dt} + X.T \quad \text{or} \quad T.X' = 2X.T' + X.T \quad \text{or} \quad T\frac{X'}{X} = 2\frac{T'}{T} + 1 = c \quad \text{(say)}$$

(*a*) $\frac{X'}{X} = c$ or $\frac{1}{X}\frac{dX}{dx} = c$ or $\frac{dX}{X} = c\,dx$

On integration $\log X = cx + \log a$. or $\log \frac{X}{a} = cx$ or $\frac{X}{a} = e^{cx}$ or $X = ae^{cx}$

(*b*) $\frac{2T'}{T} + 1 = c$ or $\frac{T'}{T} = \frac{1}{2}(c-1)$ or $\frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(c-1)$ or $\frac{dT}{T} = \frac{1}{2}(c-1)\,dt$

On integration $\log T = \frac{1}{2}(c-1)t + \log b$ or $\log \frac{T}{b} = \frac{1}{2}(c-1)t$

or $$\frac{T}{b} = e^{\frac{1}{2}(c-1)t} \quad \text{or} \quad T = be^{\frac{1}{2}(c-1)t}$$

Putting the value of $X$ and $T$ in (2), we have

$$u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$$

or $$u = ab\, e^{cx + \frac{1}{2}(c-1)t} \qquad \text{...(3)}$$

or $u(x, 0) = ab\, e^{cx}$

But $u(x, 0) = 6\, e^{-3x}$

*i.e.* $ab\, e^{cx} = 6\, e^{-3x}$ or $ab = 6$ and $c = -3$

Putting the value of $ab$ and $c$ in (3), we have

$$u = 6\, e^{-3x + \frac{1}{2}(-3-1)t}$$

$$u = 6\, e^{-3x - 2t}$$

which is the required solution. **Ans.**

**Example 2.** *Use the method of separation of variables to solve the equation :*

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

*given that $v = 0$ when $t \to \infty$, as well as $v = 0$ at $x = 0$ and $x = l$.*

**Solution.** $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \qquad ...(1)$$

Let us assume that $v = XT$ where $X$ is a function of $x$ only and $T$ that of $t$ only.

$$\frac{\partial v}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Substituting these values in (1) we get

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

Let each side of (2) be equal to a constant $(-p^2)$

or $$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \qquad ...(2)$$

$$\frac{1}{T}\frac{dT}{dt} = -p^2 \quad \text{or} \quad \frac{dT}{dt} + p^2T = 0 \qquad ...(3)$$

and $$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \quad \text{or} \quad \frac{d^2X}{dx^2} + p^2X = 0 \qquad ...(4)$$

Solving (3) and (4) we have

$$T = C_1\, e^{-p^2 t}$$

and, $$X = C_2 \cos px + C_3 \sin px$$

$\therefore$ $$v = C_1\, e^{-p^2 t}(C_2 \cos px + C_3 \sin px) \qquad ...(5)$$

Putting $x = 0$, $v = 0$ in (5), we get

$$0 = C_1\, e^{-p^2 t}\, C_2 \qquad \therefore C_2 = 0, \text{ since } C_1 \neq 0$$

On putting the value of $C_2$ in (5), we get

$$v = C_1\, e^{-p^2 t}\, C_3 \sin px \qquad ...(6)$$

Again putting $x = l$, $v = 0$ in (6) we get

$$0 = C_1\, e^{-p^2 t} \,.\, C_3 \sin pl$$

Since $C_3$ cannot be zero.

$\therefore$ $\sin pl = 0 = \sin n\pi$ $\therefore$ $p = \frac{n\pi}{l}$, $n$ is any integer.

On putting the value of $p$ in (6) it becomes

Hence $$v = C_1\, C_3\, e^{-\frac{n^2\pi^2 t}{l^2}} \sin\frac{n\pi x}{l}$$

$$v = b_n\, e^{\frac{-n^2\pi^2 t}{l^2}} \sin\frac{n\pi x}{l} \qquad b_n = C_1\, C_2$$

This equation satisfies the given condition for all integral values of $n$. Hence taking $n = 1, 2, 3, \ldots\ldots\ldots$, *the most general solution is*

$$v = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2\pi^2 t}{l^2}} \frac{\sin n\pi x}{l} \qquad \textbf{Ans.}$$

## Exercise 9.15

Using the method of separation of variables, find the solution of the following equations

**1.** $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = c\,x^{\frac{k}{2}}\,y^{\frac{k}{3}}$

**2.** $\dfrac{\partial u}{\partial x} + u = \dfrac{\partial u}{\partial t}$ if $u = 4e^{-3x}$ when $t = 0$ **Ans.** $u = 4e^{-3x-2t}$

**3.** $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ when $x = 0.$ **Ans.** $u = e^{2x-5y}$

**4.** $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u$, $u = 3e^{-x} - e^{-5x}$ at $t = 0$

(*A.M.I.E.T.E, Winter 2002, 2000*) **Ans.** $u = 3e^{t-x} - e^{2t-5x}$

**5.** $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$; $u(x, 0) = 4e^{-x}$ (*A.M.I.E.T.E., Summer 2000*) **Ans.** $u = 4e^{-x+3/2\,y}$

**6.** $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2(x+y)u$ **Ans.** $u = ce^{x^2+y^2+k(x-y)}$

**7.** $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ If $u(x, 0) = 4x - \dfrac{1}{2}x^2$ **Ans.** $u = \left(4x - \dfrac{x^2}{2}\right)e^{-p^2 t}$

**8.** $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t}$ if $u(x, 0) = x(4-x)$ **Ans.** $u = x(4-x)\,e^{-\frac{p^2 t}{2}}$

**9.** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = 2x$ when $0 \le x \le \dfrac{l}{2}$

$= 2(l-x)$ when $\dfrac{l}{2} \le x \le l$

**Ans.** $u = 2xe^{-k^2 t}$ for $0 \le x \le \dfrac{l}{2}$, $u = 2(l-x)e^{-k^2 t}$ for $\dfrac{l}{2} \le x \le l.$

**10.** $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \sin \pi x$ **Ans.** $u = \sin \pi x \,.\, e^{-p^2 t}$

**11.** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ if $u(x, 0) = x^2(25 - x^2)$ **Ans.** $u = x^2(25 - x^2)\,e^{-p^2 t}$

**12.** $x^2 u_{xx} + 3y^2 u = 0$

**13.** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = c_1 e^{[1+\sqrt{(1-p^2)}]x + p^2 y} + c_2 e^{[1-\sqrt{1-p^2}]x + p^2 y}$

**14.** $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ If $u(x, 0) = \dfrac{1}{2}x(1-x)$ **Ans.** $u = \dfrac{x}{2}(1-x)\cos pt + c_2 \sin pt\,(c_3 \cos px + c_4 \sin px)$

**15.** $16\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ if $u(x, 0) = x^2(5-x)$ **Ans.** $u = x^2(5-x)\cos pt + c_4 \sin pt\left(c_1 \cos\dfrac{px}{4} + c_2 \sin\dfrac{px}{4}\right)$

**16.** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$ if $u = 0$ **Ans.** $u = (c_1 \cos px + c_2 \sin px)\,c_3 e^{-(p^2+2)y}$

**17.** $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$ **Ans.** $u = A\,e^{1/2\,(x^2 - y^2)\,k}$

**18.** $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$, **Ans.** $u = A\,e^{k(x+y)}$

**19.** $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u$, $u(x, 0) = 4e^{-3x}$ (*A.M.I.E.T.E., Summer 2001*) **Ans.** $u = 4e^{-(3x+2y)}$

**20.** $2\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} + 5u = 0$, $u(0, y) = 2e^{-y}$ **Ans.** $u = 2e^{-x-y}$