21.1 GAMMA FUNCTION

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that* $\Gamma 1 = 1$

Solution. $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Put $n = 1$, $\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$ **Proved**

Example 2. *Prove that*

(*i*) $\overline{|n+1} = n\,\overline{|n}$ (*ii*) $\overline{|n+1} = \underline{|n}$ (*Reduction formula*)

Solution.

(*i*) $\overline{|n} = \int_0^\infty x^{n-1} e^{-x}\, dx$ ...(1)

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \ldots + \frac{x^n}{\underline{|n}} + \ldots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$\therefore$ $\Gamma n = (n-1)\,\overline{|n-1}$ ...(2)

$\overline{|n+1} = n\,\Gamma n$ Replacing n by $(n+1)$ **Proved**

(*ii*) Replace n by $n-1$ in (2), we get

$$\overline{|n-1} = (n-2)\,\overline{|n-2}$$

Putting the value $\Gamma n-1$ in (2), we get

$$\Gamma n = (n-1)(n-2)\,\overline{|n-2}$$

Similarly $\Gamma n = (n-1)(n-2)\ldots 3.2.1\,\Gamma 1$ ... (3)

Putting the value of $\overline{|1}$ in (3), we have

$$\Gamma n = (n-1)(n-2)\ldots 3.2.1.1$$

$$\Gamma n = \underline{|n-1}$$

Replacing n by $n+1$, we have

$\overline{|n+1} = \underline{|n}$ **Proved**

Example 3. *Evaluate* $\int_0^{\infty} \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^{\infty} x^{1/4} e^{-\sqrt{x}}\, dx$ ...(1)

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,t\,dt$ in (1), we get

$$I = \int_0^{\infty} t^{1/2} e^{-t}\, 2\,t\,dt = 2\int_0^{\infty} t^{3/2} e^{-t}\, dt$$

$$= 2\left\lceil\frac{5}{2}\right. \quad \text{By definition}$$

$$= 2\,.\,\frac{3}{2}\left\lceil\frac{3}{2}\right. = 2\,.\,\frac{3}{2}\,.\,\frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{3}{2}\sqrt{\pi}$$ **Ans.**

Example 4. *Evaluate* $\int_0^{\infty} \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let $I = \int_0^{\infty} \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$ ...(1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3\,t^2\,dt$ in (1) we get

$$I = \int_0^{\infty} t^{3/2} e^{-t}\, 3\,t^2\, dt = 3\int_0^{\infty} t^{7/2} e^{-t}\, dt = 3\left\lceil\frac{9}{2}\right. = 3\,.\,\frac{7}{2}\,.\,\frac{5}{2}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{315}{16}\sqrt{\pi}$$ **Ans.**

Example 5. *Evaluate* $\int_0^{\infty} x^{n-1} e^{-h^2x^2}\, dx.$

Solution. Let $I = \int_0^{\infty} x^{n-1} e^{-h^2x^2}\, dx$ ...(1)

Putting $t = h^2x^2$ or $x = \frac{\sqrt{t}}{h}$ or $dx = \frac{dt}{2\,h\sqrt{t}}$,

(1) becomes
$$I = \int_0^{\infty} \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\frac{dt}{2\,h\,\sqrt{t}}$$

$$= \frac{1}{2\,h^n}\int_0^{\infty} t^{\frac{n-1}{2}} e^{-t}\frac{dt}{\sqrt{t}} = \frac{1}{2\,h^n}\int_0^{\infty} t^{\frac{n-2}{2}} e^{-t}\, dt$$

$$= \frac{1}{2\,h^n}\left\lceil\frac{n}{2}\right.$$ **Ans.**

Example 6. *Evaluate* $\int_0^{\infty} \frac{x^a}{a^x}\, dx.$ $(a > 1)$

Solution: $I = \int_0^{\infty} \frac{x^a}{a^x}\, dx$ ...(1)

Putting $a^x = e^t$ or $x \log a = t$, $x = \frac{t}{\log a}$, $dx = \frac{dt}{\log a}$ in (1), we have

$$I = \int_0^{\infty} \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^{\infty} e^{-t}\, t^a\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\left\lceil a+1\right.$$ **Ans.**

Example 7. *Evaluate* $\int_0^1 x^{n-1}\,.\left[\log_e.\left(\frac{1}{x}\right)\right]^{m-1}.\,dx$

Solution: Put $\log_e\frac{1}{x} = t$ or $x = e^{-t}$ $\therefore dx = -e^{-t}\,dt$

$$\int_0^1 x^{n-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} dx = \int_\infty^0 (e^{-t})^{n-1}[t]^{m-1}(-e^{-t}\,dt) = \int_0^\infty e^{-nt}\,t^{m-1}\,dt$$

Put $nt = u$ or $t = \frac{u}{n}$ $\quad\therefore\quad dt = \frac{du}{n}$

$$= \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1}\frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}\,u^{m-1}\,du = \frac{1}{n^m}\Gamma m$$ **Ans.**

21.2 TRANSFORMATION OF GAMA FUNCTION

Prove that (1) $\int_0^\infty e^{-ky}y^{n-1}\,dy = \frac{\Gamma n}{k^n}$ (2) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (3) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy = \Gamma n$

Solution: We know that $\Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx$ ...(1)

(i) Replace x by $k\,y$, so that $dx = kdy$; then

(1) becomes $\Gamma n = \int_0^\infty (ky)^{n-1}e^{-ky}\,k\,dy.$

$$\Gamma n = k^n\int_0^\infty e^{-ky}y^{n-1}\,dy$$

$\therefore$ $$\int_0^\infty e^{-ky}y^{n-1}\,dy = \frac{\Gamma n}{k^n}$$ ...(2) **Proved**

(*ii*) Replace x^n by y, $n\,x^{n-1}\,dx = dy$ in (1), then

$$\Gamma n = \int_0^\infty y^{\frac{n-1}{n}}\,e^{-y^{1/n}}\frac{dy}{nx^{n-1}}$$

$$= \int_0^\infty y^{\frac{n-1}{n}}\,e^{-y^{1/n}}\frac{dy}{ny^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\,dy$$

When $n = \frac{1}{2,}$ $$\Gamma\frac{1}{2} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}\,dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$$

Proved $$\Gamma\frac{1}{2} = \sqrt{\pi}$$

(*iii*) Substitute e^{-x} by y, $-e^{-x}\,dx = dy$

$-x = \log y$, $x = \log\frac{1}{y}$, Then (1) becomes

$$\Gamma n = -\int_1^0\left(\log\frac{1}{y}\right)^{n-1} y\,.\frac{dy}{e^{-x}}$$

$$= \int_0^1\left(\log\frac{1}{y}\right)^{n-1} y\,.\frac{dy}{y} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1} dy.$$ **Proved**

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma -\frac{1}{2}$ (*ii*) $\Gamma\frac{-3}{2}$ (*iii*) $\Gamma\frac{-15}{2}$ (*iv*) $\Gamma\frac{7}{2}$ (*v*) $\Gamma 0$

Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\frac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\,e^{-x}\,dx$ **Ans.** $\Gamma\frac{3}{2}$ **3.** $\int_0^\infty x^4 e^{-x^2}\,dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}\,dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

5. $\int_0^{\infty}\int_0^{\infty} e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1}\,dx\,dy,\ a, b, m, n > 0$ **Ans.** $\dfrac{\Gamma m \Gamma n}{4\,a^m b^n}$

6. $\int_0^1 \left(\log\dfrac{1}{y}\right)^{n-1} dy,\quad n > 0$ **Ans.** Γn

7. $\int_0^1 \dfrac{dx}{\sqrt{-\log x}}$ **Ans.** $\sqrt{\pi}$

8. $\int_0^1 (x\log x)^3\,dx$ **Ans.** $-\dfrac{3}{128}$

9. $\int_0^1 \dfrac{dx}{\sqrt{x\log\dfrac{1}{x}}}$ **Ans.** $\sqrt{2\pi}$

10. Prove that $1.3.5.....(2n-1) = \dfrac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

11. $\int_0^{\infty} e^{-y^{1/m}}\,dy = m\Gamma m.$
