FOURIER SERIES

PART-5

12.11. PARSEVAL'S FORMULA

$$\int_{-c}^{c} [f(x)]^2\, dx = c\left\{\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)\right\}$$

Solution. We know that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$ ... (1)

Multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} a_n f(x) \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n f(x) \sin\frac{n\pi x}{c} \quad ...(2)$$

Integrating term by term from $-c$ to c, we have

$$\int_{-c}^{c} [f(x)]^2\, dx = \frac{a_0}{2}\int_{-c}^{c} f(x)\, dx + \sum_{n=1}^{\infty} a_n \int_{-c}^{c} f(x) \cos\frac{n\pi x}{c}\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\, dx \quad ...(3)$$

In article 12.10, we have the following results

$$\int_{-c}^{c} f(x)\, dx = c\, a_0$$

$$\int_{-c}^{c} f(x) \cos\frac{n\pi x}{c}\, dx = c\, a_n$$

$$\int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}\, dx = c\, b_n$$

On putting these integrals in (3), we get

$$\int_{-c}^{c} [f(x)]^2\, dx = c\,\frac{a_0^2}{2} + \sum_{n=1}^{\infty} c\, a_n^2 + \sum c\, b_n^2 = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

This is the Parseval's formula

Note. 1. If $0 < x < 2c$, then $\int_{0}^{2c} [f(x)]^2\, dx = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$

2. If $0 < x < c$ (Half range cosine series), $\int_{0}^{c} [f(x)]^2 = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$

3. If $0 < x < c$ (Half range sine series), $\int_{0}^{c} [f(x)]^2\, dx = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} b_n^2\right]$

4. R.M.S. $= \left\{\frac{\int_{a}^{b} [f(x)]^2\, dx}{b-a}\right\}^{\frac{1}{2}}$

Example 20. *By using the sine series for $f(x) = 1$ in $0 < x < \pi$ show that*

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$$ *(Hamirpur 1996)*

Solution. sine series is $f(x) = \Sigma\, b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}[(-1)^n - 1]$$

$$= \frac{4}{n\pi} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if } n \text{ is even}$$

Then, the sine series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \ldots\ldots$$

$$\int_0^c [f(x)]^2\,dx = \frac{c}{2}\left[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + \ldots\ldots\right]$$

$$\int_0^\pi (1)^2\,dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + \ldots\ldots\right]$$

$$[x]_0^\pi = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots$$

Proved.

Example 21. *If* $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

using half range cosine series, show that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$$

Solution. Half range cosine series is

$$f(x) = \frac{a_0}{2} + \Sigma\, a_n \cos\frac{n\pi x}{c}$$

where

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \pi$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \pi(2-\pi)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \pi\left[x\,\frac{\sin\frac{\pi x}{2}}{\frac{n\pi}{2}} - \left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \pi\left[(2-x)\,\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right] = \frac{4}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$a_1 = 0,\ a_2 = \frac{-4}{\pi},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{-4}{9\pi} \ldots\ldots$$

$$\int_0^c [f(x)]^2\, dx = \frac{c}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\right]$$

$$\int_0^1 (\pi\, x)^2\, dx + \int_1^2 \pi^2 (2-x)^2\, dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\,\pi^2} + \ldots\ldots\right]$$

$$\pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\,\pi^2} + \ldots\ldots$$

$$\frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + \ldots\ldots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$$

Ans.

Example 22. *Prove that for $0 < x < \pi$*

$$(a)\quad x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right]$$

$$(b)\quad x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\ldots\right]$$

Deduce from (a) and (b) respectively that

$$(c)\ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad (d)\ \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi}{945}$$

Solution. Half range cosine series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) = \frac{2}{\pi}\left[\frac{\pi\, x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\,\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - \frac{\pi(-1)^n}{n^2} + 0 - \frac{\pi}{n^2}\right] = \frac{2}{\pi}\left(\frac{\pi}{n^2}\right)\left[-(-1)^n - 1\right]$$

$$= \frac{-4}{n^2} \qquad \text{when } n \text{ is even}$$

$$= 0 \qquad \text{when } n \text{ is odd}$$

Hence,

$$x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2\, dx = \frac{a_0^2}{2} + \Sigma\, a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} (\pi^2 x^2 - 2\pi\, x^3 + x^4)\, dx = \frac{1}{2}\left(\frac{\pi^4}{9}\right) + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2 x^3}{3}-\frac{2\pi x^4}{4}+\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{3}-\frac{2\pi^5}{4}+\frac{\pi^5}{5}\right] = \frac{\pi^4}{18}+\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18}+\sum_{n=1}^{\infty}\frac{1}{n^4} \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

(*b*) Half range sine series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\frac{(-1)^n}{n^3}+\frac{2}{n^3}\right] = \frac{4}{\pi n^3}\left[-(-1)^n+1\right]$$

$$= \frac{8}{n^3\pi} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even}$$

$$\therefore \quad x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\ldots\ldots\right]$$

By Parseval's formula

$$\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)^2\,dx = \Sigma\, b_n^2$$

$$\frac{\pi^2}{15} = \frac{64}{\pi^2}\left[\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\ldots\right]$$

$$\frac{\pi^4}{960} = \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}$$

$$\text{Let } S = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\ldots\ldots = \left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\ldots\ldots\right)+\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\ldots\ldots\right)$$

$$S = \frac{\pi^4}{960}+\left(\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\ldots\right) = \frac{\pi^4}{960}+\frac{1}{2^6}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\ldots\right]$$

$$S = \frac{\pi^4}{960}+\frac{S}{64}$$

$$S-\frac{S}{64} = \frac{\pi^4}{960} \quad \text{or} \quad \frac{63s}{64} = \frac{\pi^4}{960}$$

$$S = \frac{\pi^4}{960}\times\frac{64}{63} = \frac{\pi^4}{945}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^4}{945}$$ **Proved.**

Exercise 12.5

1. Prove that in $0 < x < c$,

$$x = \frac{c}{2}-\frac{4c}{\pi^2}\left(\cos\frac{\pi x}{c}+\frac{1}{3^2}\cos\frac{3\pi x}{c}+\frac{1}{5^2}\cos\frac{5\pi x}{c}+\ldots\ldots\right)$$

and deduce that

(*i*) $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots = \frac{\pi^4}{96}$ (*ii*) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots\ldots = \frac{\pi^4}{90}$

12.12. FOURIER SERIES IN COMPLEX FORM

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + ... + a_n \cos\frac{n\pi x}{l} + ...$$

$$+ b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + ... + b_n \sin\frac{n\pi x}{l} + ... \quad ...(1)$$

We know that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

On putting the values of $\cos x$ and $\sin x$ in (1), we get

$$f(x) = \frac{a_0}{2} + a_1 \frac{e^{\frac{i\pi x}{l}} + e^{\frac{-i\pi x}{l}}}{2} + a_2 \frac{e^{\frac{2i\pi x}{l}} + e^{\frac{-2i\pi x}{l}}}{2} + ... + b_1 \frac{e^{\frac{i\pi x}{l}} - e^{\frac{i\pi x}{l}}}{2i} + b_2 \frac{e^{\frac{2i\pi x}{l}} - e^{\frac{-2i\pi x}{l}}}{2i} + ...$$

$$= \frac{a_0}{2} + (a_1 - ib_1) e^{\frac{i\pi x}{l}} + (a_2 - ib_2) e^{\frac{2i\pi x}{l}} + ... + (a_1 + ib_1) e^{\frac{-i\pi x}{l}} + (a_2 + ib_2) e^{\frac{-2i\pi x}{l}} + ...$$

$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + ... + c_{-1} e^{\frac{-i\pi x}{l}} + c_{-2} e^{\frac{2i\pi x}{l}} + ...$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{l}}$$

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

where $c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\, dx$

$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right] \Rightarrow c_n = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right\}dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{in\pi x}{l}}\,dx$$

Example 23. *Obtain the complex form of the Fourier series of the function*

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution. $c_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-inx}\,dx + \int_0^{\pi} 1\cdot e^{-inx}\,dx\right] = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= -\frac{1}{2n\pi i}\left[e^{-in\pi} - 1\right] = -\frac{1}{2n\pi i}[\cos n\pi - 1] = -\frac{1}{2n\pi i}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \frac{1}{in\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + ...\right] + \frac{1}{i\pi}\left[\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + ...\right]$$

$$= \frac{1}{2} - \frac{1}{i\pi}\left[\left(e^{ix} - e^{-ix}\right) + \frac{1}{3}\left(e^{3ix} - e^{-3ix}\right) + \frac{1}{5}\left(e^{5ix} - e^{-5ix}\right) + ...\right]$$ **Ans.**

Exercise 12.6

Find the complex form of the Fourier series of

1. $f(x) = e^{-x}, \ -1 \le x \le 1.$ **Ans.** $\sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2} \sinh 1 \cdot e^{in\pi x}$

2. $f(x) = e^{ax}, \ -l < x < l$ **Ans.** $\frac{2}{\pi} - \frac{2}{\pi}\left[\frac{e^{2it} + e^{-2it}}{1 \cdot 3} + \frac{e^{4it} + e^{-4it}}{3 \cdot 5} + \frac{e^{6it} + e^{-6it}}{5 - 7} + \ldots\right]$

3. $f(x) = \cos ax, \ -\pi < x < \pi$ **Ans.** $\frac{a}{\pi} \sin a\pi + \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2 - n^2}$