FOURIER SERIES

PART-4

12.10 CHANGE OF INTERVAL AND FUNCTIONS HAVING ARBITRARY PERIOD

In electrical engineering problems, the period of the function is not always 2π but T or $2c$. This period must be converted to the length 2π. The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined in the interval $(-c, c)$. Now we want to change the function to the period of 2π so that we can use the formulae of a_n, b_n as discussed in article 12.6.

$\because$ $2c$ is the interval for the variable x.

$\therefore$ 1 is the interval for the variable $= \dfrac{x}{2c}$

$\therefore$ 2π is the interval for the variable $= \dfrac{x\,2\pi}{2c} = \dfrac{\pi x}{c}$

so put $\qquad z = \dfrac{\pi x}{c} \quad$ or $\quad x = \dfrac{zc}{\pi}$

Thus the function $f(x)$ of period $2c$ is transformed to the function

$f\left(\dfrac{cz}{\pi}\right)$ or $F(z)$ of period 2π.

$F(z)$ can be expanded in the Fourier series.

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + b_1 \sin z + b_2 \sin 2z + \ldots$$

where
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} F(z)\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)dz$$

$$= \frac{1}{c}\int_0^{2c} f(x)\, d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)dx \quad \text{put } z = \frac{\pi x}{c}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(z)\cos nz\,dz = \frac{1}{\pi}\int_0^{2\pi} f\left(\frac{cz}{\pi}\right)\cos nx\,dz$$

$$= \frac{1}{\pi}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\, d\left(\frac{\pi x}{c}\right) = \frac{1}{c}\int_0^{2c} f(x)\cos\frac{n\pi x}{c}\,dx. \qquad \left[\text{Put } z = \frac{\pi x}{c}\right]$$

Similarly, $\displaystyle b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}\,dx.$

Cor. Half range series [Interval $(0, c)$]

Cosine series:

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + a_n\cos\frac{n\pi x}{c} + \ldots$$

where
$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx, \quad a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

Sine series:

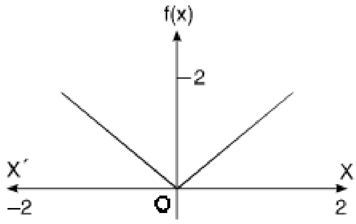
$$f(x) = b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots + b_n\sin\frac{n\pi x}{c} + \ldots$$

where
$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx.$$

Example 12. *A periodic function of period 4 is defined as*

$$f(x) = |x|, \quad -2 < x < 2.$$

Find its Fourier series expansion.



Solution.

$$f(x) = |x| \qquad -2 < x < 2$$
$$f(x) = x \qquad 0 < x < 2$$
$$= -x \qquad -2 < x < 0$$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx + \frac{1}{2}\int_{-2}^{0} (-x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 + \frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^{0} = \frac{1}{4}(4-0) + \frac{1}{4}(0+4) = 2$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$= \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] + \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$= -\frac{8}{n^2\pi^2} \qquad \text{if } n \text{ is odd.}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$b_n = 0$ as $f(x)$ is even function.

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots$$

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \ldots\right]$$ **Ans.**

Example 13. *Find Fourier half-range even expansion of the function,*

$$f(x) = (-x/l) + 1, \quad 0 \le x \le l.$$

Solution. $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l \left(-\frac{x}{l}+1\right)dx$

$$= \frac{2}{l}\left[-\frac{x^2}{2l} + x\right]_0^l = \frac{2}{l}\left[-\frac{l^2}{2l} + l\right] = \frac{2l}{l}\left[-\frac{1}{2}+1\right] = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(-\frac{x}{l}+1\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(-\frac{x}{l}+1\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \left(-\frac{1}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \frac{l}{n^2\pi^2}\cos n\pi + \frac{l}{n^2\pi^2}\right] = \frac{2}{l}\frac{l}{n^2\pi^2}[-(-1)^n + 1] = \frac{2}{n^2\pi^2}[1-(-1)^n]$$

$$= \frac{4}{n^2 \pi^2} \text{ when } n \text{ is odd.}$$

$$= 0 \quad \text{when } n \text{ is even.}$$

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l}\ \ldots\right] \qquad \textbf{Ans.}$$

Example 14. *Find the Fourier half-range cosine series of the function*

$$f(t) = 2t, \qquad 0 < t < 1$$

$$= 2(2-t), \qquad 1 < t < 2$$

(*Kuvempu 1996, A.M.I.E.T.E.,Summer 1997 1996*)

Solution.

$$f(t) = 2t, \qquad 0 < t < 1$$

$$= 2(2-t), \qquad 1 < t < 2$$

Let

$$f(t) = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{c} + a_2 \cos\frac{2\pi t}{c} + a_3 \cos\frac{3\pi t}{c} + \ldots$$

$$+ b_1 \sin\frac{\pi t}{c} + b_2 \sin\frac{2\pi t}{c} + b_3 \sin\frac{3\pi t}{c} + \ldots \qquad \ldots(1)$$

Here $c = 2$, because it is half range series.

Hence

$$a_0 = \frac{2}{c}\int_0^c f(t)\,dt = \frac{2}{2}\int_0^1 2t\,dt + \frac{2}{2}\int_1^2 2(2-t)\,dt$$

$$= \left[t^2\right]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2 = 1 + \left[(4t - t^2)\right]_1^2 = 1 + (8 - 4 - 4 + 1) = 2$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt = \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \left[2t\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_0^1$$

$$+ \left[(4-2t)\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (-2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\right] + \left[0 - \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{4}{n\pi}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1 - \frac{n\pi}{2}\sin\frac{n\pi}{2}\right]$$

If $n = 1$, $\quad a_1 = \frac{8}{\pi^2}\left[0 - 1 - \frac{\pi}{2}\right] = -\frac{8}{\pi^2} - \frac{4}{\pi}.$

If $n = 2$, $\quad a_2 = \frac{8}{4\pi^2}[-1 - 1] = -\frac{16}{4\pi^2} = -\frac{4}{\pi^2}$

If $n = 3$, $\quad a_3 = \frac{8}{9\pi^2}\left[0 - 1 + \frac{3\pi}{2}\right] = -\frac{8}{9\pi^2} + \frac{4}{3\pi}$

Putting the values of $a_0, a_1, a_2, a_3 \ldots$ in (1) we get

$$f(t) = 1 - \left(\frac{8}{\pi^2} + \frac{4}{\pi}\right)\cos\frac{\pi t}{2} - \frac{4}{\pi^2}\cos\frac{2\pi t}{2} + \left(-\frac{8}{9\pi^2} + \frac{4}{3\pi}\right)\cos\frac{3\pi t}{2} + \ldots \qquad \textbf{Ans.}$$

Example 15. *Obtain the Fourier cosine series expansion of the periodic function defined by*

$$f(t) = \sin\left(\frac{\pi t}{l}\right), \quad 0 < t < l$$

Solution.

$$f(t) = \sin\left(\frac{\pi t}{l}\right), \quad 0 < t < l$$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right) dt = \frac{2}{l}\left(-\frac{l}{\pi}\cos\frac{\pi t}{l}\right)_0^l = -\frac{2}{\pi}(\cos\pi - \cos 0) = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)\cos\frac{n\pi t}{l}\,dt = \frac{1}{l}\int_0^l \left[\sin\left(\frac{\pi t}{l}+\frac{n\pi t}{l}\right) - \sin\left(\frac{n\pi t}{l}-\frac{\pi t}{l}\right)\right] dt$$

$$= \frac{1}{l}\int_0^l \sin(n+1)\frac{\pi t}{l}\,dt - \frac{1}{l}\int_0^l \sin(n-1)\frac{\pi t}{l}\,dt$$

$$= \frac{1}{l}\left[-\frac{l}{(n+1)\pi}\cos\frac{(n+1)\pi t}{l}\right]_0^l - \frac{1}{l}\left[\frac{l}{(n-1)\pi}\cos\frac{(n-1)\pi t}{l}\right]_0^l$$

$$= \frac{-1}{(n+1)\pi}[\cos(n+1)\pi - \cos 0] + \frac{1}{(n-1)\pi}[\cos(n-1)\pi - \cos 0]$$

$$= \frac{1}{(n+1)\pi}[(-1)^{n+1}-1] + \frac{1}{(n-1)\pi}[(-1)^{n+1}-1]$$

$$= (-1)^{n+1}\left[-\frac{1}{(n+1)\pi}+\frac{1}{(n-1)\pi}\right] + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi}$$

$$= (-1)^{n+1}\frac{2}{(n^2-1)\pi} - \frac{2}{(n^2-1)\pi} = \frac{2}{(n^2-1)\pi}[(-1)^{n+1}-1]$$

$$= \frac{-4}{(n^2-1)\pi} \quad \text{when } n \text{ is even}$$

$$= 0 \quad \text{when } n \text{ is odd.}$$

The above formula for finding the value of a_1 is not applicable.

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi t}{l}\cos\frac{\pi t}{l}\,dt = \frac{1}{l}\int_0^l \sin\frac{2\pi t}{l}\,dt$$

$$= \frac{1}{l}\left(-\frac{l}{2\pi}\cos\frac{2\pi t}{l}\right)_0^l = -\frac{1}{2\pi l}(\cos 2\pi - \cos 0) = 0 = \frac{1}{2\pi l}(1-1) = 0$$

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{l} + a_2\cos\frac{2\pi t}{l} + a_3\cos\frac{3\pi t}{l} + a_4\cos\frac{4\pi t}{l} + \ldots$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi t}{l} + \frac{1}{15}\cos\frac{4\pi t}{l} + \frac{1}{35}\cos\frac{6\pi t}{l} + \ldots\right]$$ **Ans.**

Example 16. *Find the Fourier series expansion of the periodic function of period 1*

$$f(x) = \frac{1}{2} + x, \quad -\frac{1}{2} < x \le 0$$

$$= \frac{1}{2} - x, \quad 0 < x < \frac{1}{2}$$

(A.M.I.E.T.E., Winter 1996)

Solution. Let
$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + \ldots + b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + \ldots \quad \ldots(1)$$

Here $2c = 1$ or $c = \frac{1}{2}$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx + \frac{1}{1/2}\int_0^{1/2}\left(\frac{1}{2}-x\right)dx$$

$$= 2\left[\frac{x}{2}+\frac{x^2}{2}\right]_{-1/2}^{0} + 2\left[\frac{x}{2}-\frac{x^2}{2}\right]_0^{1/2} = 2\left[\frac{1}{4}-\frac{1}{8}\right] + 2\left[\frac{1}{4}-\frac{1}{8}\right] = \frac{1}{2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\frac{\sin 2n\pi x}{2n\pi} - (1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\frac{\sin 2n\pi x}{2n\pi} - (-1)\left(\frac{-\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right] + 2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4n^2\pi^2}\right] = \frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2} \qquad \text{if } n \text{ is odd}$$

$$= 0 \qquad \text{if } n \text{ is even}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin\frac{n\pi x}{1/2}\,dx + \frac{1}{1/2}\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin\frac{n\pi x}{1/2}\,dx$$

$$= 2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\sin 2n\pi x\,dx + 2\int_{0}^{1/2}\left(\frac{1}{2}-x\right)\sin 2n\pi x\,dx$$

$$= 2\left[\left(\frac{1}{2}+x\right)\left(\frac{-\cos 2n\pi x}{2n\pi}\right) - (1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+ 2\left[\left(\frac{1}{2}-x\right)\left(-\frac{\cos 2n\pi x}{2n\pi}\right) - (-1)\left(-\frac{\sin 2n\pi x}{4n^2\pi^2}\right)\right]_{0}^{1/2}$$

$$= 2\left[-\frac{1}{4n\pi}\right] + 2\left[\frac{1}{4n\pi}\right] = 0$$

Substituting the values of $a_0, a_1, a_2, a_3, \ldots b_1, b_2, b_3 \ldots$ in (1) we have

$$f(x) = \frac{1}{4}+\frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2}+\frac{\cos 6\pi x}{3^2}+\frac{\cos 10\pi x}{5^2}+\ldots\right] \quad \textbf{Ans.}$$

Example 17. *Prove that*

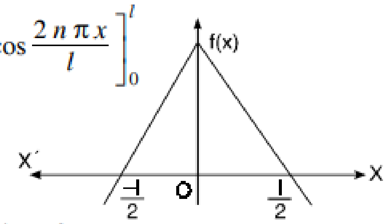
$$\frac{l}{2}-x = \frac{l}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}, \quad 0 < x < l.$$

Solution. $f(x) = \frac{1}{2}-x$

$$a_0 = \frac{1}{l/2}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)dx = \frac{2}{l}\left[\frac{lx}{2}-\frac{x^2}{2}\right]_0^l = 0$$

$$a_n = \frac{1}{l/2}\int_0^l f(x)\cos\frac{n\pi x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{l}{2}-x\right)\frac{1}{2\,n\,\pi}\sin\frac{2\,n\,\pi\,x}{l}+(-1)\frac{l^2}{4\,n^2\,\pi^2}\cos\frac{2\,n\,\pi\,x}{l}\right]_0^l$$



$$= \frac{2}{l}\left[0-\frac{l^2}{4\,n^2\,\pi^2}\cos 2\,n\,\pi+\frac{l^2}{4\,n^2\,\pi^2}\right]$$

$$= \frac{2}{l}\,\frac{l^2}{4\,n^2\,\pi^2}(-\cos 2\,n\,\pi+1) = \frac{l}{2\,n^2\,\pi^2}(-1+1) = 0$$

$$b_n = \frac{1}{l/2}\int_0^l f(x)\sin\frac{n\,\pi\,x}{l/2}\,dx = \frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\sin\frac{2\,n\,\pi\,x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{1}{2}-x\right)\left(-\frac{1}{2\,n\pi}\cos\frac{2\,n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{4n^2\pi^2}\sin\frac{2\,n\pi x}{l}\right)_0^l\,\right]$$

$$= \frac{2}{l}\left[\frac{l}{2}\,\frac{l}{2\,n\,\pi}\cos 2\,n\,\pi-0+\frac{l}{2}\cdot\frac{l}{2\,n\,\pi}(1)\right] = \frac{2}{l}\left[\frac{l^2}{2\,n\,\pi}\right] = \frac{l}{n\pi}$$

Fourier series is

$$f(x) = \frac{a_0}{2}+a_1\cos\frac{n\,\pi\,x}{l/2}+a_2\cos\frac{2\,n\,\pi\,x}{l/2}+a_3\cos\frac{3\,n\,\pi\,x}{l/2}+\ldots$$

$$+\,b_1\sin\frac{n\,\pi\,x}{l/2}+b_2\sin\frac{2\,n\,\pi\,x}{l/2}+b_3\sin\frac{3\,n\,\pi\,x}{l/2}+\ldots$$

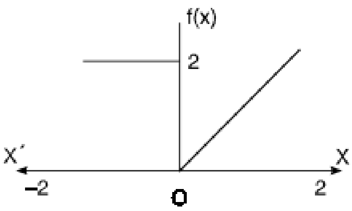
$$\frac{l}{2}-x = \frac{l}{\pi}\sin\frac{2\,\pi\,x}{l}+\frac{l}{2\,\pi}\sin\frac{4\,\pi\,x}{l}+\frac{l}{3\,\pi}\sin\frac{6\,\pi\,x}{l}+\ldots$$

$$= \frac{l}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{2\,n\,\pi\,x}{l}$$ **Proved.**

Example 18. *Find the Fourier series corresponding to the function $f(x)$ defined in $(-2,\ 2)$ as follows*

$$f(x) = 2 \quad \text{in} \quad -2 \le x \le 0$$
$$= x \quad \text{in} \quad 0 < x < 2.$$

Solution. Here the interval is $(-2, 2)$ and $c = 2$



$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2.\,dx+\int_0^2 x.\,dx\right]$$

$$= \frac{1}{2}\left[[\,2\,x\,]_{-2}^{0}+\left(\frac{x^2}{2}\right)_0^2\right] = \frac{1}{2}[4+2] = 3$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\,\pi\,x}{c}\right)dx = \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\,\pi\,x}{2}\,dx+\int_0^2 x\cos\frac{n\,\pi\,x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\,\pi}\left(\sin\frac{n\,\pi\,x}{2}\right)_{-2}^{0}+\left(x\,\frac{2}{n\,\pi}\sin\frac{n\,\pi\,x}{2}+\frac{4}{n^2\,\pi^2}\cos\frac{n\,\pi\,x}{2}\right)_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\,\pi^2}\cos n\,\pi-\frac{4}{n^2\,\pi^2}\right] = \frac{2}{n^2\,\pi^2}[\,(-1)^n-1\,]$$

$$= -\frac{4}{n^2\,\pi^2} \qquad \text{when } n \text{ is odd}$$

$$= 0 \qquad \text{when } n \text{ is even.}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) + (1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right] = \frac{1}{2}\left[-\frac{4}{n\pi}\right] = -\frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + \ldots$$

$$+ b_1\sin\frac{\pi x}{c} + b_2\sin\frac{2\pi x}{c} + b_3\sin\frac{3\pi x}{c} + \ldots$$

$$= \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\right\}$$

$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\right\}$$ **Ans.**

Example 19. *Expand $f(x) = e^x$ in a cosine series over (0, 1).*

Solution. $f(x) = e^x$ and $c = 1$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = 2(e-1)$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x + \cos n\pi x)\right]_0^1 = 2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi + \cos n\pi) - \frac{1}{n^2\pi^2+1}\right]$$

$$= \frac{2}{n^2\pi^2+1}[(-1)^n e - 1]$$

$$f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + a_3\cos 3\pi x + \ldots\ldots\ldots$$

$$e^x = e - 1 + 2\left[\frac{-e-1}{\pi^2+1}\cos\pi x + \frac{e-1}{4\pi^2+1}\cos 2\pi x + \frac{-e-1}{9\pi^2+1}\cos 3\pi x + \ldots\right]$$ **Ans.**

Exercise 12.4

1. Find the Fourier series to represent $f(x)$, where

$$f(x) = -a \quad -c < x < 0$$
$$= a \quad 0 < x < c$$

Ans. $\frac{4a}{\pi}\left[\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right]$

2 Find the half-range sine series for the function

$$f(x) = 2x - 1 \quad 0 < x < 1.$$

Ans. $-\frac{2}{\pi}\left[\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \ldots\right]$

3. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

Ans. $1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]$

4. Find the Fourier series of the function

$$f(x) = \begin{bmatrix} -2 & \text{for} & -4 < x < -2 \\ x & \text{for} & -2 < x < 2 \\ 2 & \text{for} & 2 < x < 4 \end{bmatrix}$$

Ans. $\frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \left(\frac{4}{3\pi} - \frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4} - \frac{2}{2\pi}\sin\frac{4\pi x}{4} + \ldots$

5. Find the Fourier series to represent

$$f(x) = x^2 - 2 \quad \text{from} \quad -2 < x < 2.$$

Ans. $-\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2}\ldots\right]$

6. If $f(x) = e^{-x}$ $\quad -c < x < c$, show that

$$f(x) = (e^c - e^{-c})\left\{\frac{1}{2c} - c\left(\frac{1}{c^2+\pi^2}\cos\frac{\pi x}{c} - \frac{1}{c^2+4\pi^2}\cos\frac{2\pi x}{c} + \ldots\right)\right.$$

$$\left. - \pi\left(\frac{1}{c^2+\pi^2}\sin\frac{\pi x}{c} - \frac{2}{c^2+4\pi^2}\sin\frac{2\pi x}{c}\ldots\right)\right\}$$ (*Hamirpur 1996, Mysore 1994*)

7. A sinusodial voltage E sin ωt is passed through a half wave rectifier which clips the negative portion of the wave. Develop the resulting portion of the function

$$u(t) = 0 \quad \text{when} \quad -\frac{T}{2} < t < 0$$

$$= E\sin\omega t \quad \text{when} \quad 0 < t < \frac{T}{2} \qquad \left(T = \frac{2\pi}{\omega}\right)$$ (*Mangalore 1997*)

Ans. $\frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \frac{1}{5.7}\cos 6\omega t + \ldots\right]$

8. A periodic square wave has a period 4. The function generating the square is

$$f(t) = 0 \quad \text{for} \quad -2 < t < -1$$

$$= k \quad \text{for} \quad -1 < t < 1$$

$$= 0 \quad \text{for} \quad 1 < t < 2$$

Find the Fourier series of the function. **Ans.** $f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \ldots\right]$

9. Find a Fourier series to represent x^2 in the interval $(-l, l)$. (*Nagpur 1997*)

Ans. $\frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\cos\pi x - \frac{\cos\pi x}{2^2} + \frac{\cos 3\pi x}{3^2}\right]$