

FOURIER SERIES

PART-1

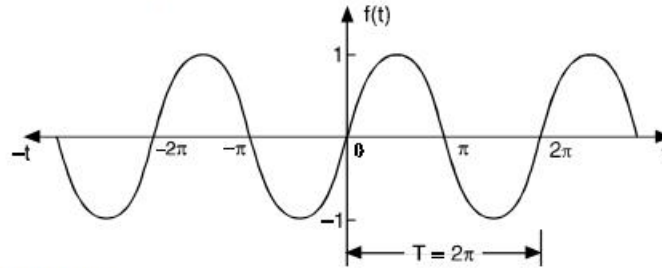
12.1 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t + T) = f(t + 2T) = \dots$ then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \dots$ so $\sin x$ is a periodic function with the period 2π . This is also called sinusoidal periodic function.



12.2 FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form.

$$\begin{aligned} & \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx + \dots \\ & \quad + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx + \dots \\ & = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \end{aligned}$$

is called the *Fourier series*, where $a_0, a_1, a_2, \dots, a_n, \dots, b_1, b_2, b_3, \dots, b_n, \dots$ are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following:

- (i) A constant term a_0 (called d.c. component in electrical work).
- (ii) A component at the fundamental frequency determined by the values of a_1, b_1 .
- (iii) Components of the harmonics (multiples of the fundamental frequency) determined by $a_2, a_3, \dots, b_2, b_3, \dots$. And $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are known as *Fourier coefficients* or Fourier constants.

12.3. DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

- (1) is single-valued (2) is bounded
- (3) has at most a finite number of maxima and minima.
- (4) has only a finite number of discontinuous
- (5) is $f(x + 2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$, then

$$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^P a_n \cos nx + \sum_{n=1}^P b_n \sin nx$$

converges to $f(x)$ as $P \rightarrow \infty$ at values of x for which $f(x)$ is continuous and to $\frac{1}{2} [f(x+0) + f(x-0)]$ at points of discontinuity.

12.4. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

12.5 USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

$$(i) \int_0^{2\pi} \sin nx \, dx = 0$$

$$(ii) \int_0^{2\pi} \cos nx \, dx = 0$$

$$(iii) \int_0^{2\pi} \sin^2 nx \, dx = \pi$$

$$(iv) \int_0^{2\pi} \cos^2 nx \, dx = \pi$$

$$(v) \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$$

$$(vi) \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = 0$$

$$(vii) \int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$$

$$(viii) \int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$$

$$(ix) [uv] = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

where $v_1 = \int v \, dx$, $v_2 = \int v_1 \, dx$ and so on. $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on

(x) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$

12.6 DETERMINATION OF FOURIER COEFFICIENTS (EULER'S FORMULAE)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots \\ + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots \quad \dots(1)$$

(i) **To find a_0 :** Integrate both sides of (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) \, dx = \frac{a_0}{2} \int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x \, dx + a_2 \int_0^{2\pi} \cos 2x \, dx + \dots + a_n \int_0^{2\pi} \cos nx \, dx + \dots \\ + b_1 \int_0^{2\pi} \sin x \, dx + b_2 \int_0^{2\pi} \sin 2x \, dx + \dots + b_n \int_0^{2\pi} \sin nx \, dx + \dots$$

$$= \frac{a_0}{2} \int_0^{2\pi} dx, \text{ (other integrals = 0 by formulae (i) and (ii) of Art. 12.5)}$$

$$\int_0^{2\pi} f(x) \, dx = \frac{a_0}{2} 2\pi, \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \quad \dots(2)$$

(ii) **To find a_n :** Multiply each side of (1) by $\cos nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) \cos nx \, dx = \frac{a_0}{2} \int_0^{2\pi} \cos nx \, dx + a_1 \int_0^{2\pi} \cos x \cos nx \, dx + \dots + a_n \int_0^{2\pi} \cos^2 nx \, dx \dots \\ + b_1 \int_0^{2\pi} \sin x \cos nx \, dx + b_2 \int_0^{2\pi} \sin 2x \cos nx \, dx + \dots \\ = a_n \int_0^{2\pi} \cos^2 nx \, dx = a_n \pi$$

$$\therefore a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx \quad \dots(3)$$

By taking $n = 1, 2 \dots$ we can find the values of a_1, a_2, \dots

(iii) **To find b_n :** Multiply each side of (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\begin{aligned} \int_0^{2\pi} f(x) \sin nx dx &= \frac{a_0}{2} \int_0^{2\pi} \sin nx dx + a_1 \int_0^{2\pi} \cos x \sin nx dx + \dots + a_n \int_0^{2\pi} \cos nx \sin nx dx + \dots \\ &\quad + b_1 \int_0^{2\pi} \sin x \sin nx dx + \dots + b_n \int_0^{2\pi} \sin^2 nx dx + \dots \\ &= b_n \int_0^{2\pi} \sin^2 nx dx \\ &\quad \text{(All other integrals = 0, Article No. 12.5)} \\ &= b_n \pi \end{aligned}$$

$$\therefore b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad \dots(4)$$

Note: To get similar formula of a_0 , $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1. Find the Fourier series representing

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

$$\text{Solution. Let } f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \dots \quad \dots(1)$$

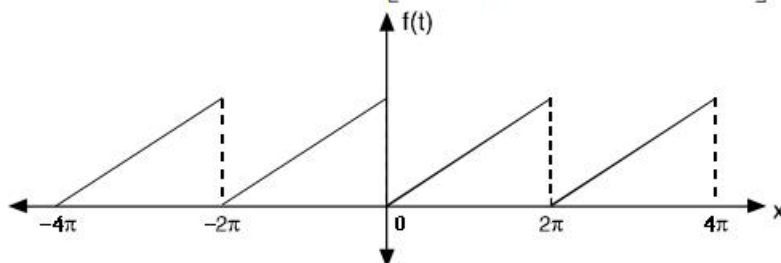
$$\text{Hence } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = 2\pi$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx \\ &= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2} \right) \right]_0^{2\pi} = \frac{1}{\pi} \left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} \right] = \frac{1}{n^2 \pi} (1 - 1) = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx \\ &= \frac{1}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - 1 \cdot \left(-\frac{\sin nx}{n^2} \right) \right]_0^{2\pi} = \frac{1}{\pi} \left[\frac{-2\pi \cos 2n\pi}{n} \right] = -\frac{2}{n} \end{aligned}$$

Substituting the values of $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ in (1)

$$x = \pi - 2 \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right] \quad \text{Ans.}$$



Example 2. Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$.

$$\text{Deduce that } \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

(U.P. II Semester, Summer 2003)

Solution. Let $x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$... (1)

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3} \right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \int_{-\pi}^{\pi} (x + x^2) \cos nx dx$$

$$= \frac{1}{\pi} \left[(x + x^2) \frac{\sin nx}{n} - (2x + 1) \frac{(-\cos nx)}{n^2} + (2) \left(-\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[(2\pi + 1) \frac{\cos n\pi}{n^2} - (-2\pi + 1) \frac{\cos(-n\pi)}{n^2} \right] = \frac{1}{\pi} \left[4\pi \frac{\cos n\pi}{n^2} \right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx dx$$

$$= \frac{1}{\pi} \left[(x + x^2) \left(-\frac{\cos nx}{n} \right) - (2x + 1) \left(-\frac{\sin nx}{n^2} \right) + 2 \frac{\cos nx}{n^3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[-(\pi + \pi^2) \frac{\cos n\pi}{n} + 2 \frac{\cos n\pi}{n^3} + (-\pi + \pi^2) \frac{\cos n\pi}{n} - 2 \frac{\cos n\pi}{n^3} \right]$$

$$= \frac{1}{\pi} \left[-\frac{2\pi}{n} \cos n\pi \right] = -\frac{2}{n} (-1)^n$$

Substituting the values of a_0, a_n, b_n in (1) we get

$$x + x^2 = \frac{\pi^2}{3} + 4 \left[-\cos x + \frac{1}{2^2} \cos 2x - \frac{1}{3^2} \cos 3x + \dots \right]$$

$$- 2 \left[-\sin x + \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x + \dots \right] \dots (2)$$

Put $x = \pi$ in (2), $\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$... (3)

Put $x = -\pi$ in (2), $-\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$... (4)

Adding (3) and (4) $2\pi^2 = \frac{2\pi^2}{3} + 8 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$

$$\frac{4\pi^2}{3} = 8 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{Ans.}$$

Exercise 12.1

1. Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$.

Ans. $2 \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots + \frac{1}{n} \sin nx + \dots \right]$

2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \quad (\text{Mysore 1997, Osmania 1995})$$

3. Find a Fourier series to represent: $f(x) = x \sin x$, for $0 < x < 2\pi$.

Ans. $-1 + \pi \sin x - \frac{1}{2} \cos x + 2 \left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 3x}{3^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \dots \right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$.

$$\text{Ans. } \frac{2 \sinh \pi}{\pi} \left[\left(\frac{1}{2} - \frac{1}{1^2+1} \cos x + \frac{1}{2^2+1} \cos 2x - \frac{1}{3^2+1} \cos 3x + \dots \right) + \frac{1}{1^2+1} \sin x - \frac{2}{2^2+1} \sin 2x + \frac{3}{3^2+1} \sin 3x \dots \right] \frac{\pi}{\sinh \pi} = 1 + 2 \left[-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \dots \right]$$

5. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 \leq x < 2\pi$. (Nagpur 1997)

$$\text{Ans. } \frac{1 - e^{-2\pi}}{\pi} \left[\frac{1}{2} + \frac{1}{2} \cos x + \frac{1}{5} \cos 2x + \frac{1}{10} \cos 3x + \frac{1}{2} \sin x + \frac{2}{5} \sin 2x + \frac{3}{10} \sin 3x + \dots \right]$$

6. If $f(x) = \left(\frac{\pi-x}{2}\right)^2$, $0 < x < 2\pi$, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ (Madras 1998)

7. Prove that $x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$, $-\pi < x < \pi$.

Hence show that (i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ (Madras 1997, Mangalore 1997, Warangal 1996)

(ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (Mangalore 1997) (iii) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ (Madras 1997)

8. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \frac{(3x^2 - 6x\pi + 2\pi^2)}{12}$.

Prove that $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ and hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$