21.11 ERROR FUNCTION

1. $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ is called error function of x and is also written as $erf\,(x)$.

2. $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ is called complementary error function of x and is also written as $erf_c\,(x)$.

3. Important formula.

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

Example 29. *Prove that* $erf(0) = 0$

Solution.
$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$
$$erf(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}\,dt = 0$$ **Proved**

Example 30. *Prove that* $erf(\infty) = 1$

Solution. $erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

$$erf\,(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$ **Proved**

Example 31. *Prove that* $erf\,(x) + erf_c\,(x) = 1$

Solution.
$$erf\,(x) + erf_c\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$
$$= \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-t^2}\,dt + \int_x^\infty e^{-t^2}\,dt\right]$$
$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1$$ **Proved**

Example 32. *Prove that* $erf\,(-x) = -erf\,(x)$

Solution.
$$erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$
$$erf\,(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}\,dt$$ Put $t = -\mu$
$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,(-du) = -\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = -erf\,(x)$$
Proved

Example 33. *Show that*

$$\int_a^b e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}[erf\,(b) - erf\,(a)]$$

Solution. $\frac{\sqrt{\pi}}{2}[erf\,(b) - erf\,(a)]$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^b e^{-t^2}\,dt - \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}\,dt\right] = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt$$

$$= \int_0^b e^{-t^2}\,dt + \int_a^0 e^{-t^2}\,dt = \int_a^b e^{-t^2}\,dt = \int_a^b e^{-x^2}\,dx \qquad \textbf{Proved}$$

Example 34. *Show that*

$$\int_0^\infty e^{-x^2-2bx}\,dx = \frac{\sqrt{\pi}}{2} e^{b^2}\,[1 - erf(b)]$$

Solution. $\int_0^\infty e^{-x^2-2bx}\,dx$

$$= \int_0^\infty e^{-x^2-2bx-b^2+b^2}\,dx = \int_0^\infty e^{-(x+b)^2}\,.\,e^{b^2}\,dx$$

$$= e^{b^2}\left[\int_b^\infty e^{-t^2}\,dt + \int_0^b e^{-t^2}\,dt\right] \qquad [\text{Put } x + b = t]$$

$$= e^{b^2}\left[-\int_b^0 e^{-t^2}\,dt + erf(\infty)\right]$$

$$= e^{b^2}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} erf(b)\right] = e^{b^2}\frac{\sqrt{\pi}}{2}[1 - erf(b)] \qquad \textbf{Proved}$$

Example 35. *Prove that*

$$\frac{d}{dx}[erf_c(\alpha x)] = \frac{-2a}{\sqrt{\pi}} e^{-a^2x^2}$$

Solution. $\frac{d}{dx}\left[erf_c(\alpha x)\right] = \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}}\int_{\alpha x}^\infty e^{-t^2}\,dt\right]$

On applying the rule of differentiation under integral sign, we get

$$= \frac{2}{\sqrt{\pi}}\left[\int_{\alpha x}^\infty\left(\frac{\partial}{\partial x}e^{-t^2}\right)dt + \frac{d}{dx}(\infty)\,e^{-\infty} - \frac{d}{dx}(\alpha x)\,e^{-\alpha^2x^2}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + 0 - \alpha\,.\,e^{-\alpha^2x^2}\right] = -\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2} \qquad \textbf{Proved}$$

Exercise 21.5

Prove that

1. $erf_c(x) + erf_c(-x) = 2$

2. $erf_c(-x) = 1 + erf(x)$

3. $erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{1}{2!}\cdot\frac{x^5}{5} - \frac{1}{3!}\cdot\frac{x^7}{7} + \ldots\right]$

4. $\int_0^\infty e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}[1 - erf(a)]$

5. $\int_0^t erf_c(ax)\,dx = t\,erf_c(at) - \frac{e^{-a^2t^2}}{a\sqrt{\pi}} + \frac{1}{a\sqrt{\pi}}$

6. $\frac{d}{dx}[erf(ax)] = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$

7. $\frac{d}{dx}[erf(\sqrt{x})] = \frac{e^{-x}}{\sqrt{\pi x}}$

8. $\frac{d}{dx}[erf(x)] = \frac{2}{\sqrt{\pi}}e^{-x^2}$