21.3 BETA FUNCTION

$$\int_0^\infty x^{l-1}(1-x)^{m-1}\,dx \qquad (l>0,\ m>0)$$

is called the Beta function of *l, m*. It is also written as

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

21.4 EVALUATION OF BETA FUNCTION

$$\beta(l\,,m) = \frac{\Gamma l\,\Gamma m}{\Gamma (l+m)}$$

Solution. We have $\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx = \int_0^1 (1-x)^{m-1}x^{l-1}\,dx$

Integrating by parts, we have

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right)dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2}x^l\,dx$$

Again integrating by parts

$$= \frac{(m-1)(m-2)}{l(l+1)}\int_0^1 (1-x)^{m-3}x^{l+1}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2}\,dx$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1$$

$$= \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots .(l+m-2)(l+m-1)}$$

$$= \frac{|\underline{m-1}}{l(l+1)\ldots(l+m-2)(l+m-1)} \times \frac{(l-1)(l-2)\ldots 1}{(l-1)(l-2\ldots 1}$$

$$= \frac{|\underline{m-1}\ |\underline{l-1}}{1.2\ldots(l-2)(l-1).l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$= \frac{|\underline{l-1}\ |\underline{m-1}}{|\underline{l+m-1}}$$

$$= \frac{\Gamma l\,\Gamma m}{\Gamma (l+m)}$$

And if only l is positive integer and not m then

$$\beta(l, m) = \frac{|\underline{l-1}}{m(m+1)\ldots(m+l-1)}$$ **Ans.**

21.5 A PROPERTY OF BETA FUNCTION

$$\beta(l, m) = \beta(m, l)$$

Solution. We have

$$\beta(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{l-1} [1-(1-x)]^{m-1} dx$$

$$= \int_0^1 (1-x)^{l-1} x^{m-1} dx$$

$$= \int_0^1 x^{m-1} (1-x)^{l-1} dx = \beta(m, l)$$ l and m are interchanged. **Proved**

Example 8. *Evaluate* $\int_0^1 x^4 (1-\sqrt{x})^5 dx$

Solution. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2\,tdt$

$$\int_0^1 x^4 (1-\sqrt{x})^5 dx = \int_0^1 (t^2)^4 (1-t)^5 (2\,tdt)$$

$$= 2\int_0^1 t^9 (1-t)^5 dt = 2\,\beta(10, 6) = 2\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\frac{\lfloor 9 \lfloor 5}{\lfloor 15}$$

$$= 2 \cdot \frac{\lfloor 5}{10\times 11\times 12\times 13\times 14\times 15} = \frac{2\times 1\times 2\times 3\times 4\times 5}{10\times 11\times 12\times 13\times 14\times 15}$$

$$= \frac{1}{11\times 13\times 7\times 15} = \frac{1}{15015}$$ **Ans.**

Example 9. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}} dx$

Solution. Let $x^3 = y$ or $x = y^{1/3}$ or $dx = \frac{1}{3} y^{-\frac{2}{3}} dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}} dx = \int_0^1 (1-y)^{-\frac{1}{2}} \left(\frac{1}{3} y^{-\frac{2}{3}} dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}} (1-y)^{-\frac{1}{2}} dy = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3}\frac{\Gamma\frac{1}{3}\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

21.6 TRANSFORMATION OF BETA FUNCTION

We know that

$$\beta(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1} dx$$

Putting $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2} dy$ and $1 - x = \frac{y}{1+y}$.

$$\beta(l, m) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{l-1} \left(\frac{y}{1+y}\right)^{m-1} \left[-\frac{1}{(1+y)^2} dy\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}} dy$$

Since l, m can be interchanged in $\beta(l, m)$,

$$\beta(l, m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}} dy \quad \text{or} \quad \beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}} dx$$

Example 10. *Evaluate* $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Solution. We know that

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m, n)$$

$\Rightarrow$ $$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m, n) \qquad ...(1)$$

Consider $\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ $\left(\text{Put } x = \frac{1}{t}\right)$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\,dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{m-1}\frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n}(t+1)^{m+n}}\,dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Putting the value of $\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ in (1) we get

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \beta\ (m, n)$$

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta\ (m, n)$$ **Ans.**
