Electric field intensity and potential

part-2

Maxwell's first law:

Statement: The following Electrostatic Field equations will be developed in this section:

Integral form

$$\oint_{\text{Surface}} \boldsymbol{D} \bullet d\boldsymbol{a} = \int_{\text{Volume}} \rho \, dv.$$

Differential forms

$$\nabla \bullet \boldsymbol{D} = \rho.$$

$$\text{div } \boldsymbol{D} = \rho$$

Maxwell's first equation is based on Gauss' law of electrostatics published in 1832, wherein Gauss established the relationship between static electric charges and their accompanying static fields.

The above integral equation states that the electric flux through a closed surface area is equal to the total charge enclosed.

The differential form of the equation states that the divergence or outward flow of electric flux from a point is equal to the volume charge density at that point.

Divergence:

The divergence represents the volume density of the outward fluxof a vector field from an infinitesimal volume around a given point.

The following properties can all be derived from the ordinary differentiation rules of calculus. Most importantly, the divergence is a linear operator, i.e.

$$\operatorname{div}(a\mathbf{F} + b\mathbf{G}) = a \operatorname{div} \mathbf{F} + b \operatorname{div} \mathbf{G}$$

for all vector fields **F** and **G** and all real numbers a and b.

There is a product rule of the following type: if φ is a scalar-valued function and **F** is a vector field, then

$$\operatorname{div}(\varphi\mathbf{F}) = \operatorname{grad} \varphi \cdot \mathbf{F} + \varphi \operatorname{div} \mathbf{F},$$

or in more suggestive notation

$$\nabla \cdot (\varphi\mathbf{F}) = (\nabla\varphi) \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F}).$$

Another product rule for the cross product of two vector fields **F** and **G** in three dimensions involves the curl and reads as follows:

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \mathbf{G} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G},$$

or

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

The Laplacian of a scalar field is the divergence of the field's gradient:

$$\operatorname{div}(\nabla\varphi) = \Delta\varphi.$$

The divergence of the curl of any vector field (in three dimensions) is equal to zero:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Poisson’s and Laplace’s Equations:

For electrostatic field, we have seen that

$$\nabla \cdot \vec{D} = \rho_v$$

$$\vec{E} = -\nabla V \quad \text{...(53)}$$

Form the above two equations we can write

$$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v \quad \text{..(54)}$$

Using vector identity we can write, $\varepsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \varepsilon = -\rho_v$ ....(55)

For a simple homogeneous medium, ε is constant and $\nabla \varepsilon = 0$. Therefore,

$$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad \text{...............(56)}$$

This equation is known as Poisson’s equation. Here we have introduced a new operator ∇^2, (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \quad \text{...............(57)}$$

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon} \quad \text{..............(58)}$$

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{...............(59)}$$

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{...............(60)}$$

At points in simple media, where no free charge is present, Poisson’s equation reduces to

$$\nabla^2 V = 0 \quad \text{...................................(61)}$$

Which is known as Laplace’s equation.

Laplace’s and Poisson’s equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found hroughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

Solutions to Laplace's Equation in CartesianCoordinates:

Having investigated some general properties of solutions to Poisson's equation, it is now appropriate to study specific methods of solution to Laplace's equation subject to boundary conditions. Exemplified by this and the next section are three standard steps often used in representing EQS fields. First, Laplace's equation is set up in the coordinate system in which the boundary surfaces are coordinate surfaces. Then,

the partial differential equation is reduced to a set of ordinary differential equations by separation of variables. In this way, an infinite set of solutions is generated. Finally, the boundary conditions are satisfied by superimposing the solutions found by separation of variables.

In this section, solutions are derived that are natural if boundary conditions are stated along coordinate surfaces of a Cartesian coordinate system. It is assumed that the fields depend on only two coordinates, x and y, so that Laplace's equation is (Table I)

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \qquad (1)$$

This is a partial differential equation in two independent variables. One time-honored method of mathematics is to reduce a new problem to a problem previously solved. Here the process of finding solutions to the partial differential equation is reduced to one of finding solutions to ordinary differential equations. This is accomplished by the *method of separation of variables*. It consists of assuming solutions with the special space dependence

$$\Phi(x,y)=X(x)Y(y) \qquad (2)$$

In (2), X is assumed to be a function of x alone and Y is a function of y alone. If need be, a general space dependence is then recovered by superposition of these special solutions. Substitution of (2) into (1) and division by Φ then gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} \qquad (3)$$

Total derivative symbols are used because the respective functions X and Y are by definition only functions of x and y.

In (3) we now have on the left-hand side a function of x alone, on the right-hand side a function of y alone. The equation can be satisfied independent of x and y only if each of these expressions is constant. We denote this "separation" constant by k^2, and it follows that

$$\frac{d^2X}{dx^2}=-k^2X \qquad (4)$$

and

$$\frac{d^2Y}{dy^2}=k^2Y \qquad (5)$$

These equations have the solutions

$$X\sim\cos kx \quad \text{or} \quad \sin kx \qquad (6)$$
$$Y\sim\cosh ky \quad \text{or} \quad \sinh ky \qquad (7)$$

If $k = 0$, the solutions degenerate into

$$X\sim \text{constant} \quad \text{or} \quad x \qquad (8)$$
$$Y\sim \text{constant} \quad \text{or} \quad y \qquad (9)$$

The product solutions, (2), are summarized in the first four rows of Table 5.4.1. Those in the right-hand column are simply those of the middle column with the roles of x and y interchanged. Generally, we will leave the prime off the k' in writing these solutions. Exponentials are also solutions to (7). These, sometimes more convenient, solutions are summarized in the last four rows of the table.

Electric dipole:

The name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P, at which we want to know the electric and potential fields

Dipole moment:

A stronger mathematical definition is to use vector algebra, since a quantity with magnitude and direction, like the dipole moment of two point charges, can be expressed in vector form

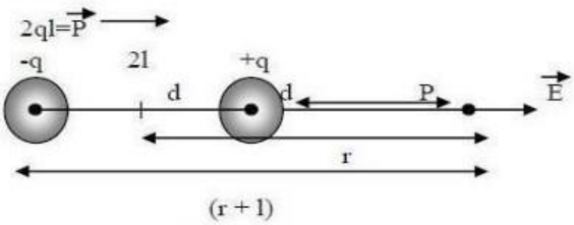
$$\mathbf{p} = q\mathbf{d}$$

Where d is the displacement vectorpointing from the negative charge to the positive charge. The electric dipole moment vector p also points from the negative charge to the positive charge.

EFI due to an electric dipole:

To calculate electric field created by a dipole on the axial line (on the same line joining the two charges),

- All the measurement of distances are to be taken from the centre(O).
- Let the distance between O to +q and O to –q be 'l'. So, total length between +q and –q will be '2l'.
- Take a point 'p' on the axial line at the distance 'r' from the centre as shown in figure.



Now, we wish to calculate electric field at point 'P'.

By using the formula for electric field due to point charge,

Electric field due to +q = $\frac{+1}{4\pi\varepsilon_0}\frac{q}{(r-l)^2}$

The distance between (P and +q) = (r-l)

Electric field due to -q = $\frac{-1}{4\pi\varepsilon_0}\frac{q}{(r+l)^2}$

The distance between (P and -q) = (r + l)

(Electric field due to +q will be positive and electric field due to- q will be negative)

Since electric field is a vector quantity so, the net electric field will be the vector addition of the two.

So, the net electric field E = $E_1 + E_2$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r-l)^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{(r+l)^2}$$

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right]$$

On solving the equation we get –

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{(r+l)^2 - (r-l)^2}{(r-l)^2(r+l)^2}\right]$$

$$E = \frac{q}{4\pi\varepsilon_0}\frac{4rl}{(r^2-l^2)^2} \text{ ......(1)}$$

We know that the dipole moment or effectiveness of dipole (P) is given by –

$$P = 2ql$$

Therefore, putting this value in eq(1), we get

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2Pr}{(r^2-l^2)^2} \text{ ......(2)}$$

Certain assumptions are made based on this equation –

Since, the dipole is very small so 'l' is also very small as compared to the distance 'r'.

So, on neglecting 'r' with respect to 'l' we get –

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2Pr}{r^4} \text{ (from eq(2))}$$

$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{2P}{r^2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2P}{r^2}$$

Note – Electric field on the axial line of dipole is not 0. Its **magnitude** is resultant as expressed above.

Torque:

An object with an electric dipole moment is subject to a torque τ when placed in an external electric field. The torque tends to align the dipole with the field. A dipole aligned parallel to an electric field has lower potential energy than a dipole making some angle with it. For a spatially uniform electric field **E**, the torque is given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}\,,$$

where p is the dipole moment, and the symbol "×" refers to the vector cross product. The field vector and the dipole vector define a plane, and the torque is directed normal to that plane with the direction given by the right-hand rule.

A dipole oriented co- or anti-parallel to the direction in which a non-uniform electric field is increasing (gradient of the field) will experience a torque, as well as a force in the direction of its dipole moment. It can be shown that this force will always be parallel to the dipole moment regardless of co- or anti-parallel orientation of the dipole.

Torque on an Electric dipole in an electric field:

Let us assume an electric dipole is placed in a uniform magnetic field as shown in figure. Each charge of dipole experience a force qE in electric field. Since points of action of these forces are different, these equal and anti paralel forces give rise to a couple that rotate the dipole and make the dipole to align in the direction of field.
The torque τ experienced by the dipole is (qE)×(2dsinθ), where 2d is the length of dipole and θ is the angle between dipole and field direction.

$$\tau = qE \times 2d\sin\theta = (q \times 2d) \times E\sin\theta = p \times E\sin\theta = \boldsymbol{p} \times \boldsymbol{E}$$

we have used the definition of dipole moment p = q×2d in the above equation. **p** and **E** are vectors representing the dipole moment and Electric field respectively.
Last step shown above is the cross product of two vectors