Electromagnetic Induction

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Contents

1 Changing Electric and Magnetic Field

1.1 Motional emf

When charge moves there acts the Lorenz force on them

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{1.1}$$

The charges move according the force experienced by them. The motion of the charges generates a current and thus an emf on the medium in which the charges are moving. This is called motional emf.

Lorentz force law is very useful to determine the direction of the induced current. To calculate the magnitude we need to know the Faraday's law of induction.

1.2 Faraday's Law

When the magnetic flux ($\Phi = \int \mathbf{B} \cdot \mathbf{dS}$) somehow changes in a closed conducting system (be it a closed loop of wire or something a bend and closed rod) there creates an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{1.2}$$

The significance of the negative sign is, the induced emf tries to oppose the change of the magnetic flux.

The basic principle the Faraday's law states is: **A changing magnetic field induces an electric field.**

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{1.3}$$

This is Faraday's law, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4}$$

This equation tells us that if you change a magnetic field, you'll create an electric field. In turn, this electric field can be used to accelerate charges which, in this context, is usually thought of as creating a current in wire. The process of creating a current through changing magnetic fields is called induction.

Faraday's law tells us that if you change the magnetic flux through S then a current will flow. There are a number of ways to change the magnetic field. You could simply move a bar magnet in the presence of circuit, passing it through the surface S; or you could replace the bar magnet with some other current density, restricted to a second wire C^0, and move that; or you could keep the second wire C^0 fixed and vary the current in it, perhaps turning it on and off. All of these will induce a current in C.

However, there is then a secondary effect. When a current flows in C, it will create its own magnetic field. This induced magnetic field will always be in the direction that opposes the change. This is called Lenz's law. If you like, ”Lenz's law” is really just the minus sign in Faraday's law.

Example A metal bar of mass *m* slides frictionlessly on two parallel conducting rails a distance *l* apart (see figure below) . A resistor *R* is connected across the rails, and a uniform magnetic field **B**, pointing into the page, fills the entire region.

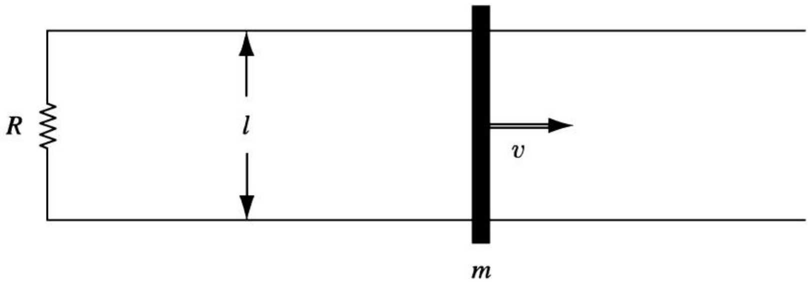

Figure 1.1: Current created by moving bar in a magnetic field

(a) If the bar moves to the right at speed *v*, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In whatdirection?

(c) If the bar starts out with speed v_0 at time t = 0, and is left to slide, what is its speed at a later time t ?

(d) The initial kinetic energy of the bar was, of course,
$\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

Solution: (a) $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv; \mathcal{E} = IR$
$\Rightarrow \quad I = \frac{Blv}{R}$. The minus sign tells you the direction of flow of the current. (**v**×**B**) is upward, in the bar, so the current is downward through the resistor.

(b) $F = IlB = \frac{B^2l^2v}{R}$, to the left.

(c) $F = ma = m\frac{dv}{dt} = -\frac{B^2l^2}{R}v \Rightarrow \frac{dv}{dt} = -\left(\frac{B^2l^2}{Rm}\right)v$
$v = v_0 e^{-\frac{B^2l^2}{mR}t}$. Hence

(d) The energy goes into heat in the resistor. The power delivered to resistor is I^2R, so

$$\frac{dW}{dt} = I^2R = \frac{B^2l^2v^2}{R^2}R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t}$$

we have taken $\alpha \equiv \frac{B^2l^2}{mR}$; So $\frac{dW}{dt} = \alpha m v_0^2 e^{-2\alpha t}$

The total energy delivered to the resistor is

$$W = \alpha m v_0^2 \int_0^\infty e^{-2\alpha t}dt = \alpha m v_0^2 \frac{e^{-2\alpha t}}{-2\alpha}\bigg|_0^\infty = \alpha m v_0^2 \frac{1}{2\alpha} = \frac{1}{2}mv_0^2$$

Example: Faraday's disk generator: A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field B , pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (See figure). Find the current in the resistor.

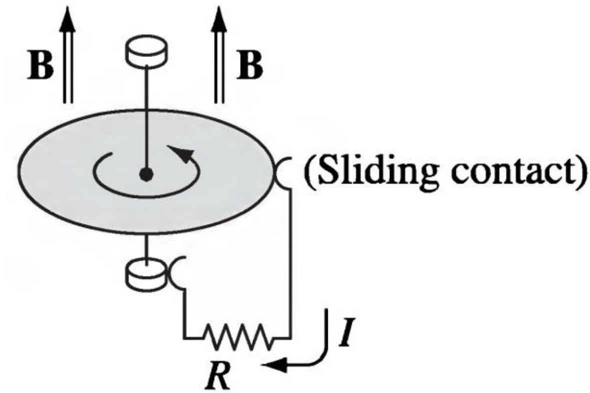

Figure 1.2: Current created by rotating metal disk

Solution: The speed of a point on the disk at a distance s from the axis is $v = \omega s$, so the force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$. The emf is therefore

$$\mathcal{E} = \int_0^a f_{\text{mag}} ds = \omega B \int_0^a s ds = \frac{\omega B a^2}{2}$$

So the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$$

The flux law or Faraday-Letz rule can also be written as in terms of electric field

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \tag{1.5}$$

Example: A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region, made by conducting material, as shown in the figure below. If $\mathbf{B}$ is changing with time, what is the induced electric field?

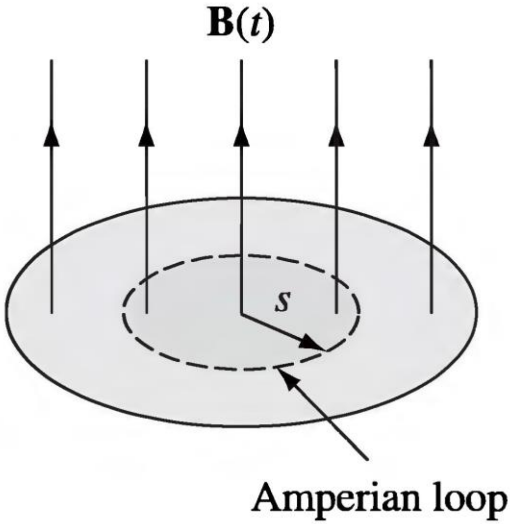

Figure 1.3: Induced Current by changing magnetic field

Solution: **E** points in the circumferential direction, just like the magnetic field inside a long straight wire carrying a uniform current density. Draw an Amperian loop of radius *s*, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}$$

Hence

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}$$

If **B** is increasing, **E** runs clockwise, as viewed from above.

Example: A line charge λ is glued onto the rim of a wheel of radius *b*,. The spokes are made of some nonconducting material. The wheel is then suspended horizontally, as shown in figure below so that it is free to rotate. In the central region, which is made by conducting material, out to radius *a,* there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

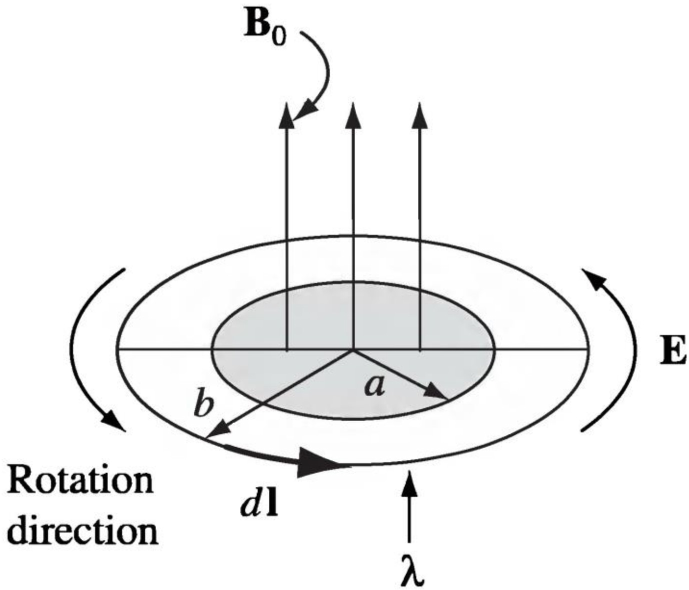

Figure 1.4: Charged disk rotates because of changing **B**

solution: The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that its field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Faraday's law, applied to the loop at radius b , says

$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\phi}$$

The torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r} \times \mathbf{F})$, or $b\lambda E$ dl. The total torque on the wheel is therefore

$$N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint dl = -b\lambda\pi a^2\frac{dB}{dt}$$

The angular momentum imparted to the wheel is

$$\int N dt = -\lambda\pi a^2 b\int_{B_0}^{0} dB = \lambda\pi a^2 b B_0$$

It doesn't matter how quickly or slowly you turn off the field; the resulting angular velocity of the wheel is the same.

Now the question is - where is the angular momentum coming from? wait for the next section.

1.3 Inductance

1.3.1 Self Inductance

Suppose that a constant current *I* flows along some curve *C.* This gives rise to a magnetic field and hence a flux $\Phi = \int_S \mathbf{B}\cdot d\mathbf{S}$ through the surface *S* bounded by *C.* Now increase the current *I.* This will increase the flux Φ . But we've just learned that the increase in flux will, in turn, induce an emf around the curve *C.* The minus sign of Lenz's law ensures that this acts to resist the change of current. The work needed to build up a current is what's needed to overcome this emf.

If a current *I* flowing around a curve *C* gives rise to a flux $\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ then the inductance *L* of the circuit is defined to be

$$L = \frac{\Phi}{I} \qquad (1.6)$$

The inductance is a property only of our choice of curve *C*.

Example: Self inductance of a Solenoid: A solenoid consists of a cylinder of length l and cross-sectional area A. We take $l \gg \sqrt{A}$ so that any end-effects can be neglected. A wire wrapped around the cylinder carries current I and winds N times per unit length. The magnetic field through the centre of the solenoid to be

$$B = \mu_0 I N \tag{1.7}$$

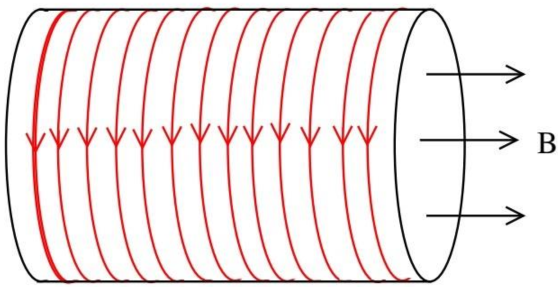

Figure 1.5:

This means that a flux through a single turn is $\Phi_0 = \mu_0 INA$. The solenoid consists of Nl turns of wire, so the total flux is

$$\Phi = \mu_0 I N^2 A l = \mu_0 I N^2 V$$

with $V = Al$ the volume inside the solenoid. The inductance of the solenoid is therefore

$$L = \mu_0 N^2 V$$

1.3.2 Mutual Inductance

Say there are two loops placed in reasonably closed distance and current I_1 passes through loop 1. The current will create magnetic field. That magnetic field will create magnetic flux around the second loop (loop 2). It can be proved that the flux flown in the second loop due to current in 1st loop is proportional to I_1.

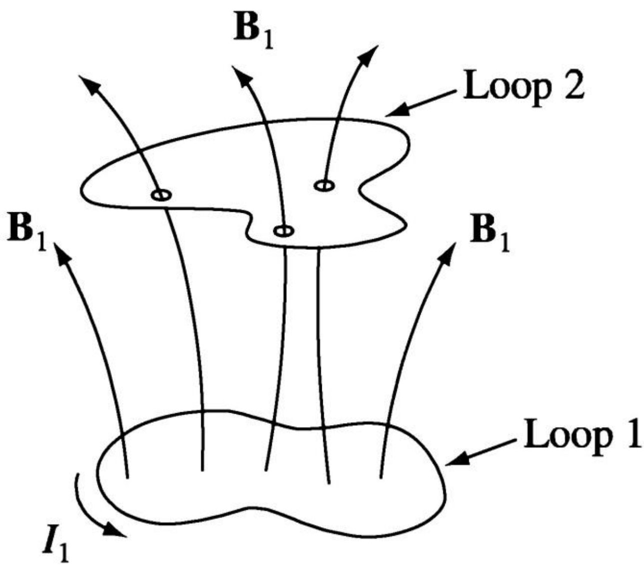

Figure 1.6: Mutual inductance

$$\Phi_2 = M_{21} I_1 \tag{1.8}$$

where M_{21} is the constant of proportionality; it is known as the mutual inductance of the two loops.

M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

If we reverse the situation: current I_2 passes through loop 2. The current will create magnetic field. That magnetic field will create magnetic flux around loop 1. It can be proved that the flux flown in the loop 1 due to current in loop 2 is proportional to I_2 with the **same proportionality constant**. i.e

$$\Phi_1 = M_{12} I_2 \tag{1.9}$$

and

$$M_{21} = M_{12} \tag{1.10}$$

Whatever the shapes and positions of the loops, the flux through loop 2 when we run a current I around loop 1 is identical to the flux through loop 1 when we send the same current I around loop 2. We drop the subscripts and call them both *M.*

Example: A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b,n_2 turns per unit length) as shown in Figure below. Current I flows in the short solenoid. What is the flux through the long solenoid? What is the mutual inductance of the system?

Solution: Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different flux through each turn of the outer solenoid. It would be a very tough task to compute the total flux this way. However, if we use the equality of the mutual inductances, the problem becomes very easy.

Just look at the reverse situation: run the current / through the outer solenoid, and calculate the flux through the inner one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I$$

This is also the flux a current I in the short solenoid would put through the long one, which is what we set out to find. Incidentally, the mutual inductance, in this case is

$$M = \mu_0 \pi a^2 n_1 n_2 l$$

1.4 Magnetostatic Energy

The definition of inductance is useful to derive the energy stored in the magnetic field. Let's take our circuit C with current I. We'll try to increase the current. The induced emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

As we mentioned above, the induced emf can be thought of as the work done in moving a unit charge around the circuit. But we have current I flowing which means that, in time δt, a charge $I\delta t$ moves around the circuit and the amount of work done is

$$\delta W = \mathcal{E} I \delta t = -LI\frac{dI}{dt}\delta t$$
$$\frac{dW}{dt} = -LI\frac{dI}{dt} = -\frac{L}{2}\frac{dI^2}{dt}$$

The work needed to build up the current is just the opposite of this. Integrating over time, we learn that the total work necessary to build up a current I along a curve with inductance L is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi$$

Following our discussion for electric energy, we identify this with the energy U stored in the system. We can write it as

$$U = \frac{1}{2}I\int_S \mathbf{B}\cdot d\mathbf{S} = \frac{1}{2}I\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S}$$
$$= \frac{1}{2}I\oint_C \mathbf{A}\cdot d\mathbf{r} = \frac{1}{2}\int d^3x\,\mathbf{J}\cdot\mathbf{A}$$

where, in the last step, we've used the fact that the current density $\mathbf{J}$ is localised on the curve C to turn the integral into one over all of space. At this point we turn to the Maxwell equation $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ to write the energy as

$$U = \frac{1}{2\mu_0}\int d^3x(\nabla \times \mathbf{B}) \cdot \mathbf{A}$$
$$= \frac{1}{2\mu_0}\int d^3x[\nabla \cdot (\mathbf{B} \times \mathbf{A}) + \mathbf{B} \cdot (\nabla \times \mathbf{A})]$$

We assume that **B** and **A** fall off fast enough at infinity so that the first term vanishes. We're left with the simple expression

$$U = \frac{1}{2\mu_0}\int d^3x \mathbf{B} \cdot \mathbf{B}$$

Combining this with electrostatic energy, we have the energy stored in the electric and magnetic fields,

$$U = \int d^3x \left(\frac{\epsilon_0}{2}\mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B} \right)$$