Rigid Body: Basic

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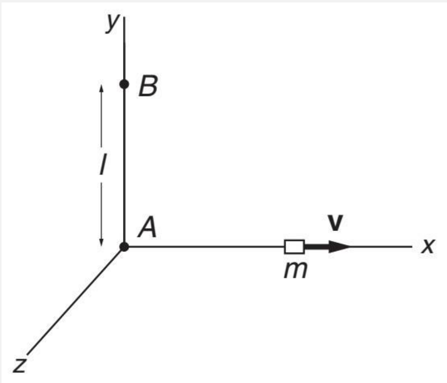
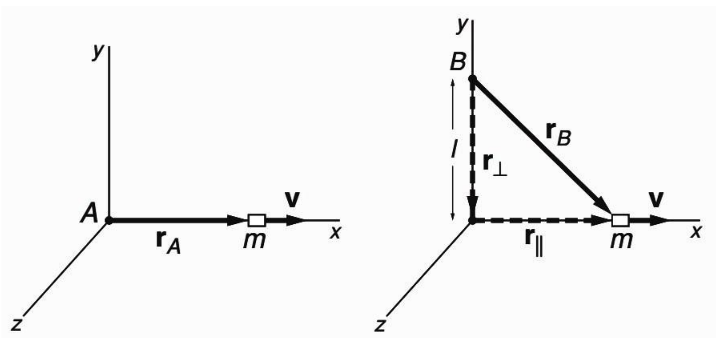
1 Basic concepts

Suppose a particle of mass m is moving in straight line with velocity v. What is its angular momentum? The most obvious answer comes to your mind is ZERO, as there is no rotation at all. Well, well did you ask - ”where is the origin and are the axes ?” Angular momentum is a vector, so it need the origin and axes before any discussion starts. And, a particle need not to be rotating to have an angular momentum.

Example 1.1: A block of mass m and negligible dimensions slides freely in the x direction with velocity $\mathbf{v} = v\,\hat{\mathbf{i}}$, as shown in the figure below. What is its angular momentum $\mathbf{L}_A$ around origin A and its angular momentum $\mathbf{L}_B$ around origin B?



Solution: Look at the picture below.



As shown in the drawing the vector from origin A to the block is $\mathbf{r}_A = x\hat{i}$. since $\mathbf{r}_A$ is parallel to $\mathbf{v}$, their cross product is zero:

$$\mathbf{L}_A = m\mathbf{r}_A \times \mathbf{v} \qquad = 0$$

Now taking origin B, we can resolve $\mathbf{r}_B$ into a component $\mathbf{r}_k$ parallel to $\mathbf{v}$ and a component $\mathbf{r}_\perp$ perpendicular to $\mathbf{v}$. Then

$$\mathbf{L}_B = m\mathbf{r}_B \times \mathbf{v} = m\left(\mathbf{r}_\parallel + \mathbf{r}_\perp\right) \times \mathbf{v}$$

With $\mathbf{r}_k \times \mathbf{v} = 0$ and $|\mathbf{r}_\perp \times \mathbf{v}| = lv\hat{\mathbf{k}}$ we have

$$\mathbf{L}_B = mlv\hat{\mathbf{k}}$$

$\mathbf{L}_B$ lies in the positive z direction because the sense of rotation is counterclockwise around the z axis.

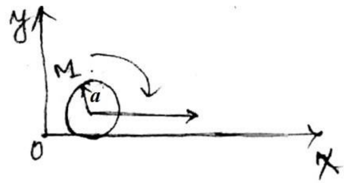
Example 1.2: A solid sphere of mass m and radius a is rolling with a linear speed v on a flat surface without slipping. The magnitude of angular momentum of the sphere in the surface is ? **[JAM 2005]**

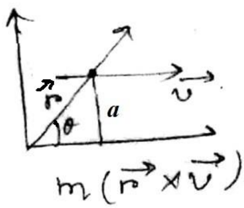
(a) $\frac{2}{5}mav$ (b) $\frac{7}{5}mav$ (c) $\frac{3}{2}mav$ (d) mav

Solution: The ans is not $\frac{2}{5}mav$ dude.

First thing first, the angular momentum of the sphere is the addition of the angular momentum of the COM of the sphere plus the angular momentum of the sphere about the center of mass.

As angular momentum of the sphere in the surface is to be calculated, we take the surface as x axis. The center of mass is moving with linear velocity v. The velocity is not on the direction of x axis, rather it is above the distance a of the surface. Now the $\vec{r}$ and the $\vec{v}$ are not in the same direction, so you need to do the cross product $\vec{r} \times \vec{v} = -rv \sin\theta \hat{z} = -av\hat{z}$





The angular momentum of the COM is then $-mav\hat{z}$

The angular momentum about the COM is simply $I\omega = \frac{2}{5}ma^2\frac{v}{a} = \frac{2}{5}mav$. the direction is $-\hat{z}$ direction is the rotation is clockwise. (you can simply use the right hand rule of cross product and should not forget that $\hat{x} \times \hat{y} = \hat{z}$). So finally the angular momentum is $-\frac{2}{5}mav\hat{z} - mav\hat{z} = -\frac{7}{5}mav\hat{z}$. The magnitude is of course $\frac{7}{5}mav$

If you are asked to find kinetic energy of the same problem? Well, you can also use the fact The kinetic energy of the sphere is the addition of the kinetic energy of the COM of the sphere plus the kinetic energy of the sphere about the center of mass.

$$\mathbf{E} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{v}{a^2} = \frac{7}{10}mv^2$$

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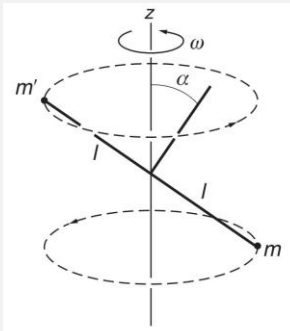
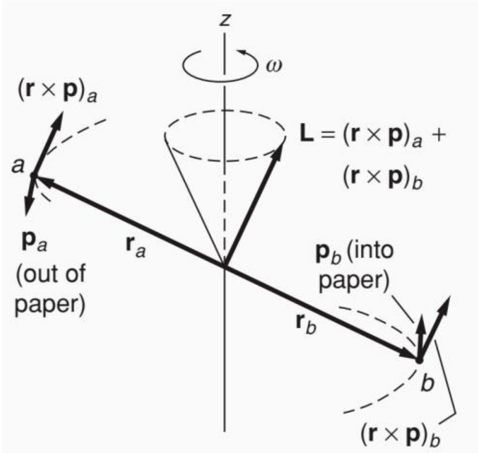
K.

Example1.3: Considerasimplerigidbodyconsisting oftwoparticlesofmassmseparatedbyamasslessrod oflength2l.Themidpointoftherodisattachedtoa verticalaxisthatrotatesatangularspeed ω aroundthe zaxis.Therodisskewedatangle α,asshowninthe sketch.Listheangularmomentumofthesystem.What willbethetorqueontherod?



(a) $\omega L \sin\alpha$ (b) $\omega g L \sin\alpha$ (c) $\omega g L \sin^2\alpha$ (d) $\frac{2}{3}\omega L \sin\alpha$

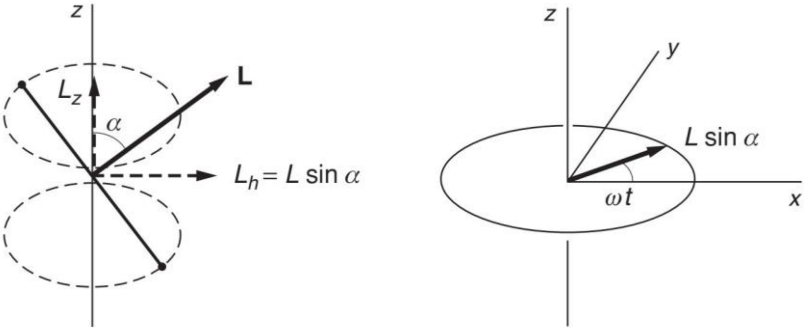
Solution: Calculate the angular momentum from the definition $\mathbf{L} = \Sigma(\mathbf{r}_i \times \mathbf{p}_i)$. See the figure below.



Let's calculate the Linear momentum first. Each mass moves in a circle of radius $l\cos\alpha$ with angular speed ω . The linear momentum of each mass is $|\mathbf{p}| = m\omega l\cos\alpha,$ and is tangential to the circular path. To calculate the angular momentum of the two masses we shall take the midpoint of the skew rod as origin, with **r** along the rod and perpendicular to **p** . We see that $|\mathbf{L}| = 2m\omega l^2 \cos\alpha.$**L** is perpendicular to the skew rod and lies in the plane of the rod and the *z* axis, as shown in the figure. **L** turns with the rod, and its tip traces out a circle about the *z* axis.

Now you may think that the angular momentum is constant then where is the torque? Well, the magnitude of the angular momentum is constant but the direction of it varies.

The torque on the rod is given by $\tau = d\mathbf{L}/dt$. We can find $d\mathbf{L}/dt$ quite easily by decomposing $\mathbf{L}$ as shown in the figure.



The component L_z parallel to the z axis, $L\cos\alpha$, is constant, so there is no torque in the z direction. The horizontal component of $\mathbf{L}$, $L_h = L\sin\alpha$, swings with the rod. If we choose $x - y$ axes so that L_h coincides with the x axis at $t = 0$, then at time t we have

$$L_x = L_h \cos\omega t \qquad = L\sin\alpha\cos\omega t$$

and

$$L_y = L_h \sin\omega t \qquad = L\sin\alpha\sin\omega t$$

Hence we get

$$\mathbf{L} = L\sin\alpha(\hat{\mathbf{i}}\cos\omega t + \hat{\mathbf{j}}\sin\omega t) + L\cos\alpha\hat{\mathbf{k}}$$

So the torque is

$$\tau = \frac{d\mathbf{L}}{dt} = L\omega \sin\alpha(-\hat{\mathbf{i}}\sin\omega t + \hat{\mathbf{j}}\cos\omega t)$$

Using $L = 2ml^2\omega \cos\alpha$ we now get

$$\tau_x = -2ml^2\omega^2 \sin\alpha \cos\alpha \sin\omega t$$
$$\tau_y = 2ml^2\omega^2 \sin\alpha \cos\alpha \cos\omega t$$

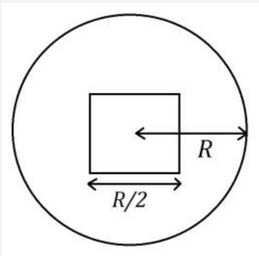
$$\tau = \sqrt{\tau_x^2 + \tau_y^2} \quad = 2ml^2\omega^2 \sin\alpha \cos\alpha \quad = \omega L \sin\alpha$$

2 Moment of Inertia

You should know moment of inertia of some simple objects which is found in any textbook. I ask you a different question.

Example2.4: Considerauniformthincirculardisk ofradius R andmass M .Aconcentricsquareofside $\frac{R}{2}$ iscutoutfromthedisk(seefig.).Whatismoment ofinertiaoftheresultantdiskaboutanaxispassing throughthecenterofthediskandperpendiculartoit?

[JAM 2017]



(a) $I = \frac{MR^2}{4}[1 - \frac{1}{48\pi}]$ (b) $I = \frac{MR^2}{2}[1 - \frac{1}{48\pi}]$

(c) $I = \frac{MR^2}{4}[1 - \frac{1}{24\pi}]$ (d) $I = \frac{MR^2}{2}[1 - \frac{1}{24\pi}]$

Solution: For this of problems you need to calculate the MOI of the whole system assuming that there is no hole or cut out element. Then you calculate the MOI of the hole (cut out portion). Then you subtract the MOI of cut out part from the MOI of the whole system. You must be cautious about the axis.

As in this particular problem the axis is same for the two. It is not necessary that all problems will have the same axis for the total and the cut cut part. In that case you need to apply the parallel axis theorem. It is also to be remembered that the density of the material is same for the two.

The MOI of the full disk is

$$\frac{1}{2}MR^2$$

. Take the density of the material to be ρ. Then

$$\pi R^2 \rho = M$$

MOI of rectangular plate of mass M and length l_1, l_2 is

$$\frac{1}{12}(l_1^2 + l_2^2)$$

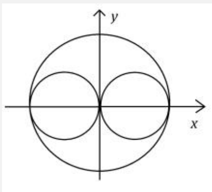
Now we use $l_1 = l_2 = R/2$, and $M = l_1 l_2 \rho = \frac{R^2}{4}\rho$. So the MOI of the plate is

$$\frac{R^2}{4}\rho\frac{1}{12}\left(\frac{R^2}{4} + \frac{R^2}{4}\right) = \frac{R^4}{96}\rho = \frac{M}{96\pi}R^2$$

Hence the MOI of the whole system is

$$\frac{1}{2}MR^2 - \frac{M}{96\pi}R^2 = \frac{1}{2}MR^2\left(1 - \frac{1}{48\pi}\right)$$

Example2.5: Themomentofinertiaofuniformradius r aboutanaxispassingthroughitscenterisgiven by $\frac{2}{5}\frac{4\pi}{3}r^5\rho$.Arigidsphereofuniformmassdensity ρ andradius R hastwosmallersphereofradius $\frac{R}{2}$ hol-lowedoutofit,asshowninfig.Themomentofinertia oftheresultingbodyaboutthe Y axisis **[GATE 2007]**



(a) $\frac{\pi\rho R^5}{4}$ (b) $\frac{5\pi\rho R^5}{12}$ (c) $\frac{7\pi\rho R^5}{12}$ (d) $\frac{3\pi\rho R^5}{4}$

Solution: MOI of the whole sphere is $\frac{2}{5}\frac{4\pi}{3}R^5\rho$, MOI of small spheres $\frac{2}{5}\frac{4\pi}{3}\frac{R}{32}\rho$ ^{5} is. This MOI is about the axis passing through the center of the small sphere. So we must shift the MOI of the small spheres to co-inside the axes. The MOI of a small sphere after using the parallel axis theorem (shift of

$R/2$) is

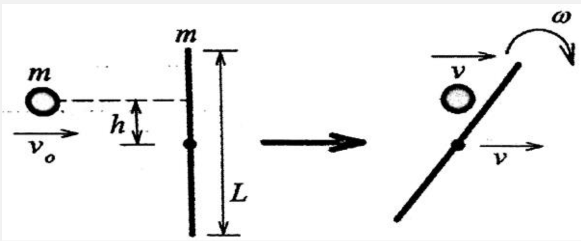
$$\frac{2}{5}\frac{4\pi}{3}\frac{R^5}{32}\rho + \frac{4\pi}{3}\frac{R^3}{8}\rho\frac{R^2}{4} = \frac{4\pi}{3}\frac{R^5}{32}\rho\left(\frac{2}{5}+1\right)$$

We have two such small sphere. So total MOI is

$$\frac{2}{5}\frac{4\pi}{3}R^5\rho - 2\frac{4\pi}{3}\frac{R^5}{32}\rho\left(\frac{2}{5}+1\right) = \frac{4\pi}{3}R^5\rho\left(\frac{2}{5}-\frac{14}{5\times 32}\right)$$

$$= \frac{5\pi R^5\rho}{12}$$

Example2.6: Amass m travelsinastraightlinewith velocity v_0 perpendiculartoauniformstickofmass m andlength L,whichisinitiallyatrest.Thedistance fromthecenterofthesticktothepathofthetraveling massis h (seefig).Nowthetravelingmass m collides elasticallywiththestick,andthecenterofthestickand themass m areobservedtomovewithequalspeedafter thecollisionAssumingthatthetravelingmass m can betreatedasapointmass,andthemomentofinertia ofthestickaboutitscenteris $I = \frac{mL^2}{12}$,itfollowsthat thedistance h mustbe **[TIFR 2010]**



(a) $\frac{L}{2}$ (b) $\frac{L}{4}$ (c) $\frac{L}{\sqrt{6}}$ (d) $\frac{L}{\sqrt{3}}$ (e) $\frac{L}{3}$ (f)zero

Solution: In these types of problems you need to apply the conservation of linear momentum and angular momentum .

And if the collision is mentioned to be elastic then you need to apply the conservation of energy also.

So first apply the conservation of linear momentum. Initially only the particle is moving with velocity v_0. not the rod. And finally both the ball and the rod is moving with same velocity v

$$mv_0 = 2mv \;.............(i)$$

The we apply angular momentum conservation. Initially the angular momentum is the angular momentum of the moving ball only. As the ball is at distance h above the COM the angular momentum is mv_0h

After the collision the ball is moving at the velocity v. So the angular momentum of the ball is mvh. And the rod is moving with angular velocity ω , so the angular momentum of the rod is $\frac{1}{12}mL^2\omega$. Applying the conservation of angular momentum we get

$$mv_0h = \frac{1}{12}mL^2\omega + mvh \;..........(ii)$$

Now as the collision is elastic we will apply the conservation of energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$
$$\implies = \frac{1}{2}\frac{1}{12}mL^2\omega^2 + mv^2 \text{ ....}(iii)$$

From these three equations you get $\omega = \frac{6v_0}{L}$, and hence $h = \frac{L}{\sqrt{6}}$

Example 2.7: A bowling ball of uniform density is thrown along a horizontal alley with initial velocity v_0 in such a way that it initially slides without rolling. The ball has mass *m*, coefficient of station μ_s and coefficient of sliding friction μ_d with the floor. Ignore the effect of air friction. Compute the velocity of the ball when it begins to roll without sliding.

Solution: When the bowling ball slides without rolling the friction $f = \mu_d mg$ gives rise to an acceleration

$$a = -\frac{f}{m} = -\mu_d g$$

The moment of *f* gives rise to an angular acceleration α given by

$$fR = \frac{2}{5}MR^2\alpha$$

as the ball has a moment of inertia $\frac{2}{5}mR^2$ about an axis through its center, R being its radius. Suppose at time t the ball begins to roll without sliding. We require

$$R\alpha t = v_0 + at$$

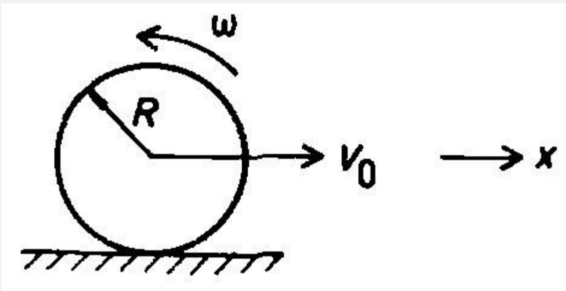
which gives

$$t = \frac{v_0}{R\alpha - a} = \frac{2mv_0}{7f} = \frac{2v_0}{7\mu_d g}$$

The velocity of the ball when this happens is

$$v = v_0 + at = v_0 - \mu_d gt = \frac{5}{7}v_0$$

Example2.8: Awheelofmass M andradius R is projectedalongahorizontalsurfacewithaninitial linearvelocity V_0 andaninitialangularvelocity ω_0 as showninFiguresoitstartsslidingalongthesurface(ω_0 tendstoproducerollinginthedirectionoppositeto V_0) Letthecoefficientoffrictionbetweenthewheelandthe surfacebe μ .



(a)Howlongisittilltheslidingceases?(b)Whatis thevelocityofthecenterofmassofthewheelatthe timewhentheslippingstops?

Solution: (a) Take the positive *x* direction as towards the right and the angular velocity $\dot{\theta}$ as positive when the wheel rotates clockwise. Assume the wheel has moment of inertia $\frac{1}{2}MR^2$ about the axle. We then have two equations of motion

$M\ddot{x} = -\mu Mg$ ....(i) $$\frac{1}{2}MR^2\ddot{\theta} = \mu MgR \text{ ....}(ii)$$

Making use of the initial conditions $\dot{x}_c = V_0, \dot{\theta}_c = -\omega_0$ at $t = 0$ we obtain by integration

$\dot{x} = V_0 - \mu gt$ ....(iii) $$\dot{\theta} = -\omega_0 + \frac{2\mu gt}{R} \text{ ....}(iv)$$

Let T be the time when sliding ceases. Then at T,

$$\dot{x} = R\dot{\theta}$$

Hence

$$V_0 - \mu gT = -R\omega_0 + 2\mu gT$$

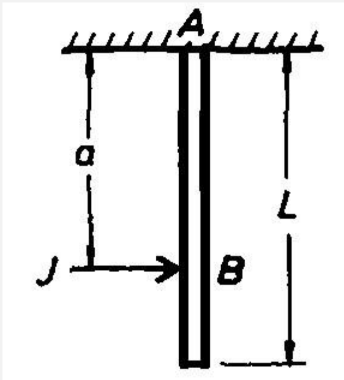
hence we get

$$T = \frac{V_0 + R\omega_0}{3\mu g}$$

(b) The velocity of the center of mass of the wheel at the time when slipping stops is

$$\dot{x} = V_0 - \mu gT = \frac{1}{3}(2V_0 - R\omega_0)$$

Example2.9: Alongthinuniformbarofmass M and length L ishungfromafixed(assumedfrictionless)axis at A asshowninFigure.Themomentofinertiaabout A is $ML^{2}/3$.



(a)Aninstantaneoushorizontalimpulse J isdelivered at B, adistanceabelow A. Whatistheinitialangular velocityofthebar?

(b)Ingeneral,asaresultof J, therewillbeanimpulse J^{C} onthebarfromtheaxisat A. Whatis J^{C}?

(c)Whereshouldtheimpulse J bedeliveredinorder that J^{C} bezero?

Solution: (a) $Ja = I(\omega - \omega_0)$, where ω_0 is the angular velocity of the bar before the impulse is delivered. As $\omega_0 = 0$, the initial angular velocity is

$$\omega = \frac{Ja}{I} = \frac{3Ja}{ML^2}$$

(b) The initial velocity of the center of mass of the bar is $v = \omega L/2$. So the change in the momentum of the bar is $Mv = M\omega L/2$. As this is equal to the total impulse on the bar, we have

$$J + J' = \frac{M\omega L}{2}$$

Hence

$$J' = \frac{M\omega L}{2} - J = J\left(\frac{3a}{2L} - 1\right)$$

(c)

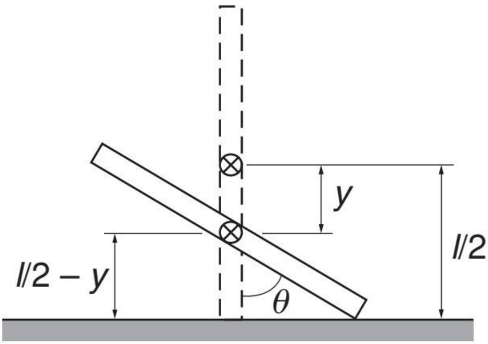
$$J' = 0, \qquad \text{if} \qquad a = \frac{2L}{3}$$

Hence there will be no impulse from the axis if J is delivered at a point $2L/3$ below A.

Example 2.10: A rod of length l and mass M, initially upright on a frictionless table, starts falling. Find the speed of the center of mass as a function of the angle θ from the vertical.

Solution: As there are no horizontal forces, the center of mass must fall straight down. Since we must find velocity as a function of position, it is natural to apply energy methods.

The figure shows the rod after it has rotated through angle θ and the center of mass has fallen distance *y.*



Initial energy is

$$\frac{mgL}{2}$$

The kinetic and potential energies after some time

$$K = \frac{1}{2} I_0 \dot{\theta}^2 + \frac{1}{2} M \dot{y}^2 ; \qquad U = Mg \left(\frac{l}{2} - y \right)$$

Because there are no dissipative forces, mechanical energy is conserved and $K + U == Mgl/2$. Hence

$$\frac{1}{2} M \dot{y}^2 + \frac{1}{2} I_0 \dot{\theta}^2 + Mg \left(\frac{l}{2} - y \right) = Mg \frac{l}{2}$$

We can eliminate $\dot{\theta}$ by using the constraint equation. From the figure you see

$$y = \frac{l}{2}(1 - \cos\theta) \quad \Longrightarrow \quad \dot{y} = \frac{l}{2}\sin\theta\dot{\theta} \quad \Longrightarrow \quad \dot{\theta} = \frac{2}{l\sin\theta}\dot{y}$$

Using $I_0 = Ml^2/12$, we obtain

$$\frac{1}{2}M\dot{y}^2 + \frac{1}{2}M\frac{l^2}{12}\left(\frac{2}{l\sin\theta}\right)^2\dot{y}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}$$

Hence

$$\dot{y}^2 = \frac{2gy}{\left[1 + 1/\left(3\sin^2\theta\right)\right]}$$

$$\dot{y} = \sqrt{\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}}$$

$$= \sqrt{\frac{3\,\mathrm{lg}(1 - \cos\theta)\sin^2\theta}{3\sin^2\theta + 1}}$$