Newtonian Mechanics: Advanced

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Contents

1 Conservation of energy in three dimension

The conservation of energy can be easily extended to three dimension. Potential energy in three dimension becomes dependent on the distance from the origin (radius vector) and it is the integration of forces with the radius vector from a reference point $\mathbf{r_0}$.

$$U(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \tag{1.1}$$

Force and displacement may not be in the same direction as shown in the figure below. In two dimension case this becomes $Fr\cos\theta$. In three dimension we need to perform line integral.

The integration of force give the difference of kinetic energy as before.

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

$$\begin{aligned}
\int_a^b \mathbf{F}\cdot d\mathbf{r} &= \int_a^b m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r} \\
&= \int_a^b m\frac{d\mathbf{v}}{dt}\cdot \mathbf{v}dt \\
&= \int_a^b \frac{m}{2}\frac{d}{dt}\left(v^2\right)dt \\
&= \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2
\end{aligned} \tag{1.2}$$

1.1 Conservative force:

A force **F** acting on a particle is conservative if and only if it satisfies two conditions: (i) F depends only on the particle's position **r** (and not on the velocity **v**, or the time *t*, or any other variable ; that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$ (ii) For any two points 1 and 2 , the work $W(1 \rightarrow 2)$ done by **F** is the same for all paths between 1 and 2 A force $\mathbf{F}(\mathbf{r})$ to be conservative a necessary and sufficient condition is to have the curl of the force to be zero.

$$\nabla \times \mathbf{F} = \mathbf{0} \tag{1.3}$$

Conservative force means the work done is path independent. We will discuss these point in details. Let us first write the force and potential energy relation in three dimension. Let us consider a particle acted on by a conservative force $\mathbf{F}(\mathbf{r})$, with corresponding potential energy $U(\mathbf{r})$, and examine the work done by $\mathbf{F}(\mathbf{r})$ in a small displacement from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$. We can evaluate this work in two ways. On the one hand, it is, by definition,

$$\begin{aligned} W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) &= \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= F_x dx + F_y dy + F_z dz \end{aligned} \tag{1.4}$$

for any small displacement $d\mathbf{r}$ with components (dx,dy,dz). On the other hand, we have seen that the work $W(\mathbf{r} \to \mathbf{r}+d\mathbf{r})$ is the same as (minus) the change in PE in the displacement:

$$\begin{aligned} W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) &= -dU = -[U(\mathbf{r} + d\mathbf{r}) - U(\mathbf{r})] \\ &= -[U(x + dx,y + dy,z + dz) - U(x,y,z)] \end{aligned} \tag{1.5}$$

We know from the definition of derivative

$$dU = U(x+dx, y+dy, z+dz) - U(x,y,z)$$
$$= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \tag{1.6}$$

So the work done is

$$W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = -\left[\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right] \tag{1.7}$$

The two expression of work in equation (1.4) and (1.7) are equivalent. So we get

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z} \tag{1.8}$$

That is, **F** is the vector whose three components are minus the three partial derivatives of U with respect to *x,y,* and *z.* A slightly more compact way to write this result is this:

$$\mathbf{F} = -\hat{\mathbf{x}}\frac{\partial U}{\partial x} - \hat{\mathbf{y}}\frac{\partial U}{\partial y} - \hat{\mathbf{z}}\frac{\partial U}{\partial z} \tag{1.9}$$

From vector calculus we know that the gradient of a scalar function is defined as

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \tag{1.10}$$

So the force is the gradient of the potential with a negative sign

$$\mathbf{F} = -\nabla U \tag{1.11}$$

This important relation gives us the force **F** in terms of derivatives of *U,* just as the definition (1.1) gave *U* as an integral of **F**. When a force **F** can be expressed in the form (4.33)*,* we say that *F* is derivable from a potential energy. Thus, we have shown that any conservative force is derivable from a potential energy.

Example 1.1: The potential energy of a certain particle is $U = Axy^2 + B\sin Cz$, where A,B and C are constants.
What is the corresponding force?

Solution: To find **F** we have only to evaluate the three partial derivatives. In doing this, you must remember that $\partial U/\partial x$ is found by differentiating with respect to *x*, treating *y* and *z* as constant, and so on. Thus $\partial U/\partial x = Ay^2$*,* and so on, and the final result is

$$\mathbf{F} = -\left(\hat{\mathbf{x}}Ay^2 + \hat{\mathbf{y}}2Axy + \hat{\mathbf{z}}BC\cos Cz\right)$$

Now we prove the statements about the conservative force stated before. Suppose a force is conservative which essentially means that the work done to move one point to another point is path independent. Look at the figure below. We do work to move particle from point *A* to *B*. Path I is the path *ACB*, Path II is the path *ADFB*. The work done is same for both paths.

$$W_{ACB} = \int_{ACB} \mathbf{F} \cdot \mathbf{dr} = W_{ADFB} = \int_{ADFB} \mathbf{F} \cdot \mathbf{dr}$$

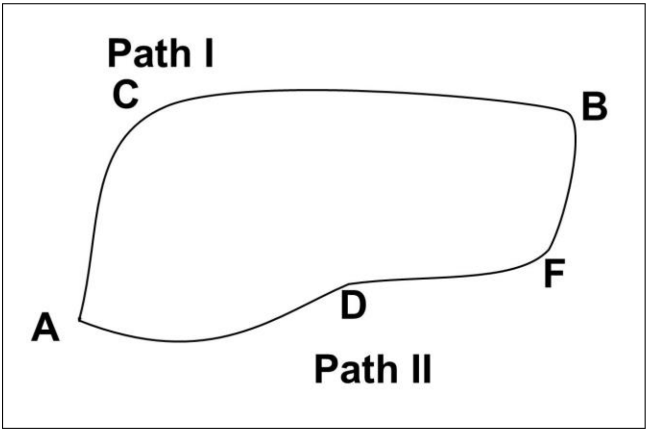

Figure 1.1:

So

$$\int_{ACB} \mathbf{F} \cdot \mathbf{dr} - \int_{ADFB} \mathbf{F} \cdot \mathbf{dr} = 0$$

Now the sign of line integral changes when we reverses the direction of integration.

$$\int_{ACB} \mathbf{F} \cdot \mathbf{dr} + \int_{BFDA} \mathbf{F} \cdot \mathbf{dr} = 0$$

hence I

$$\mathbf{F} \cdot \mathbf{dr} = 0$$

So the work done in traveling the whole path once in a closed cycle is ZERO.

From the stokes theorem we know

$$\oint \mathbf{F} \cdot \mathbf{dr} = \iint (\nabla \times \mathbf{F}) \cdot \mathbf{n} ds$$

So if the line integral of a vector around a closed curve is zero then the curl of the vector must be zero.

$$\nabla \times \mathbf{F} = 0$$

Finally we summaries our discussion of conservative force

If a force **F** is conservative in a region then all these following condition satisfies. Any one of the following five conditions implies all the others.

$\nabla \times \mathbf{F} = 0$ at every point of the region.

$\oint \mathbf{F} \cdot d\mathbf{r} = 0$ around every simple closed curve in the region.

F is conservative, that is $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration from A to B. (The path must, of course, lie entirely in the region.)

$\mathbf{F} \cdot d\mathbf{r}$ is an exact differential of a single valued function.

$\mathbf{F} = -\nabla \cdot U$, where U is a scalar potential.

Example 1.2: A particle of mass *m* moves in the *xy* plane so that its position vector is

$$\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$$

where *a,b* and ω are positive constants and $a > b$.

(a) Show that the particle moves in an ellipse. (b) Show that the force acting on the particle is always directed toward the origin.

(c) Find the kinetic energy at points *A* and *B* of the figure as shown below.

(d) Find the work done by the force field in moving theparticle from *A* to *B*.

(e) Show that the total work done by the field in movingthe particle once around the ellipse is zero.

(f) Show that the force field is conservative.

(g) Find the potential energy at points *A* and *B*. (h) Find the total energy of the particle and see that it is constant.

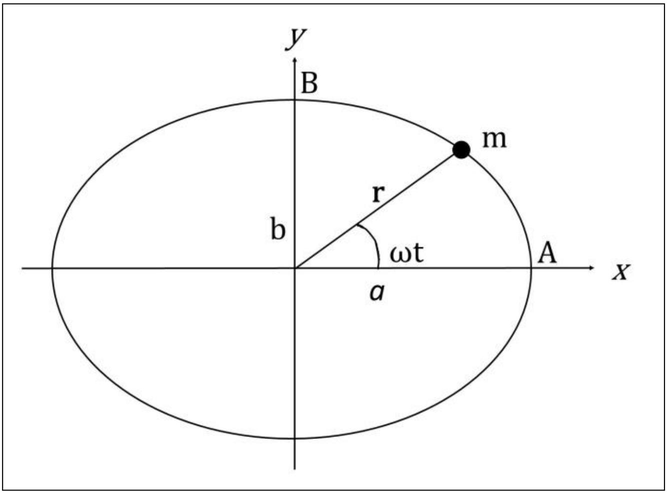

Figure 1.2:

Solution: (a) The position vector is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$$

So $x = a\cos\omega t, y = b\sin\omega t$

Then

$$(x/a)^2 + (y/b)^2 = \cos^2 wt + \sin^2 \omega t = 1$$

the ellipse is so given by $x^2/a^2 + y^2/b^2 = 1$ (b) Assuming the particle has constant mass m, the force acting on it is

$d\mathbf{v}$ $d^2\mathbf{r}$ d^2

$\mathbf{F} = m \frac{}{} = m$

$= m \; [(a\cos\omega t)\mathbf{i}$

$+ \; (b\sin\omega t)\mathbf{j}] \; dt$

$$= m\left[-\omega^2 a\cos\omega t\mathbf{i} - \omega^2 b\sin\omega t\mathbf{j}\right]$$

$dt^2 \quad dt^2$

$$= -m\omega^2[a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}] = -m\omega^2\mathbf{r}$$

which shows that the force is always directed toward the origin.

(c) Velocity

$$\mathbf{v} = d\mathbf{r}/dt = -\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j}$$

Kinetic energy

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\omega^2 a^2\sin^2\omega t + \omega^2 b^2\cos^2\omega t\right)$$

Kinetic energy at A[where $\cos\omega t = 1, \sin\omega t = 0] = \frac{1}{2}m\omega^2 b^2$

Kinetic energy at B[where $\cos\omega t = 0, \sin\omega t = 1] = \frac{1}{2}m\omega^2 a^2$

(d) Work done

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_A^B \left(-m\omega^2\mathbf{r}\right)\cdot d\mathbf{r} = -m\omega^2\int_A^B \mathbf{r}\cdot d$$
$$= -\frac{1}{2}m\omega^2\int_A^B d(\mathbf{r}\cdot\mathbf{r}) = -\frac{1}{2}m\omega^2 r^2\Big|_A^B$$
$$= \frac{1}{2}m\omega^2a^2 - \frac{1}{2}m\omega^2b^2 = \frac{1}{2}m\omega^2\left(a^2-b^2\right)$$

$\mathbf{r}$

Work done = $\frac{1}{2}m\omega^2\left(a^2-b^2\right) = \frac{1}{2}m\omega^2a^2 - \frac{1}{2}m\omega^2b^2$

= kinetic energy at *A* − kinetic energy at *B*

(e) We can find the work done by directly integrating the

$$= \int_0^{\pi/2\omega}\left[-m\omega^2(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j})\right]\cdot\left[-\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j}\right]$$
$$= \int_0^{\pi/2\omega} m\omega^3\left(a^2-b^2\right)\sin\omega t\cos\omega t dt$$
$$= \frac{1}{2}m\omega^2\left(a^2-b^2\right)\sin^2\omega t\Big|_0^{\pi/2\omega} = \frac{1}{2}m\omega^2\left(a^2-b^2\right)$$

force also. See that at *A* and *B*,*t* = 0 and *t* = *π*/2*ω* respectively. Then the Work done in moving the particle from *A* to

B

$$= \int_A^B \mathbf{F}\cdot d$$

$\mathbf{r}$

Similarly for a complete circulation around the ellipse, t goes from 0 to $t = 2\pi/\omega$.

Work done $= \int_0^{2\pi/\omega} m\omega^3 \left(a^2 - b^2\right) \sin\omega t \cos\omega t dt$

$$= \frac{1}{2} m\omega^2 \left(a^2 - b^2\right) \sin^2 \omega t \Big|_0^{2\pi/\omega} = 0$$

You can do as you did in part (d) expect the integral goes from one point to the same point, say A to A and you get zero.

$$\int_A^A \mathbf{F} \cdot d\mathbf{r} = \int_A^A \left(-m\omega^2 \mathbf{r}\right) \cdot d\mathbf{r} = -m\omega^2 \int_A^A \mathbf{r} \cdot d\mathbf{r}$$

$$= -\frac{1}{2} m\omega^2 \int_A^A d(\mathbf{r} \cdot \mathbf{r}) = -\frac{1}{2} m\omega^2 r^2 \Big|_A^A$$

$$= \frac{1}{2} m\omega^2 a^2 - \frac{1}{2} m\omega^2 a^2 = 0$$

(f)

$$F = -m\omega^2 \mathbf{r} = -m\omega^2 (x\mathbf{i} + y\mathbf{j})$$

We calculate the curl of force

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -m\omega^2 x & -m\omega^2 y & 0 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(-m\omega^2 y\right)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}\left(-m\omega^2 x\right) - \frac{\partial}{\partial x}(0)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}\left(-mw^2 y\right) - \frac{\partial}{\partial y}\left(-m\omega^2 x\right)\right]$$

$$= 0$$

As the curl of the force is zero, the force is conservative.

(g) Since the field is conservative there exists a potential V such that

$$\mathbf{F} = -m\omega^2 x\mathbf{i} - m\omega^2 y\mathbf{j} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}$$

Then

$$\partial V/\partial x = m\omega^2 x, \qquad \partial V/\partial y = m\omega^2 y, \qquad \partial V/\partial z = 0$$

from which, omitting the constant, we have

$$V = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2\left(x^2 + y^2\right) = \frac{1}{2}m\omega^2 r^2$$

which is the required potential.

(h) Kinetic energy at any point

$$T = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}m\mathbf{r}^2$$
$$= \frac{1}{2}m\left(\omega^2 a^2 \sin^2 \omega t + \omega^2 b^2 \cos^2 \omega t\right)$$

Potential energy at any point

$$V = \frac{1}{2}m\omega^2\mathbf{r}^2$$
$$= \frac{1}{2}m\omega^2\left(a^2 \cos^2 \omega t + b^2 \sin^2 \omega t\right)$$

Add and get

$$T + V = \frac{1}{2}m\omega^2\left\langle a^2 + b^2\right\rangle$$

The energy depends only on the constants *a* and *b* so the energy is constant.

Example 1.3: (a) Show that the force field $\mathbf{F}$ defined by

$$\mathbf{F} = \left\langle y^2z^3 - \mathbf{6}xz^2\right)\mathbf{i} + 2xyz^3\mathbf{j} + \left(3xy^2z^2 - 6x^2z\right)\mathbf{k}$$

is a conservative force field.
(b) Find a potential corresponding to the force.

Solution: (a) The force field *F* is conservative if and only if curl**F** = ∇ × **F** = **0**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 - 6xz^2 & 2xyz^3 & 3xy^2z^2 - 6x^2z \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}\left(3xy^2z^2 - 6x^2z\right) - \frac{\partial}{\partial z}\left(2xyz^3\right)\right]$$
$$+ \mathbf{j}\left[\frac{\partial}{\partial z}\left(y^2z^3 - 6xz^2\right) - \frac{\partial}{\partial x}\left(3xy^2z^2 - 6x^2z\right)\right]$$
$$+ \mathbf{k}\left[\frac{\partial}{\partial x}\left(2xyz^3\right) - \frac{\partial}{\partial y}\left(y^2z^3 - 6xz^2\right)\right]$$

$$= 0$$

(b) As we know the force field **F** is conservative if and only if there exists a scalar function or potential $V(x,y,z)$ such that $\mathbf{F} = -\mathrm{grad}V = -\nabla V$.

$$\mathbf{F} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}$$
$$= \left(y^2z^3 - 6xz^2\right)\mathbf{i} + 2xyz^3\mathbf{j} + \left(3xy^2z^2 - 6x^2z\right)\mathbf{k}$$

Hence if **F** is conservative we must be able to find V such that

$$\partial V/\partial x = 6xz^2 - y^2z^3$$

$$\partial V/\partial y = -2xyz^3$$

$$\partial V/\partial z = 6x^2z - 3xy^2z^2$$

To find the potential we need to integrate each component of the force and omit the common terns.

Integrate $\frac{\partial V}{\partial x}$ with respect to x keeping y and z constant. Then

$$V = 3x^2z^2 - xy^2z^3 + g_1(y,z)$$

where $g_1(y,z)$ is a function of y and z

Similarly integrate $\frac{\partial V}{\partial y}$ with respect to y (keeping x and z constant), we have

$$V = -xy^2z^3 + g_2(x,z)$$

Then integrate $\frac{\partial V}{\partial z}$ with respect to z (keeping x and y constant)

The trick is to add all three V^0s and take one term only once. $3x^2z^2$ and $-xy^2z^3$ both the terms have appeared twice. We take both the term only one time. So

the required potential is

$$V = 3x^2z^2 - xy^2z^3 + c$$

2 Collision in 2D in Center of Mass frame

2.1 Formulation of the problem

Consider two particles of masses m_1 and m_2 with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. The center of mass velocity V is

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

As shown in the part (a) of the figure below, $\mathbf{V}$ lies on the line joining $\mathbf{v}_1$ and $\mathbf{v}_2$ The velocities in the COM system are

$$\begin{aligned}\mathbf{v}_{1c} &= \mathbf{v}_1 - \mathbf{V} \\ &= \frac{m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2)\end{aligned}$$

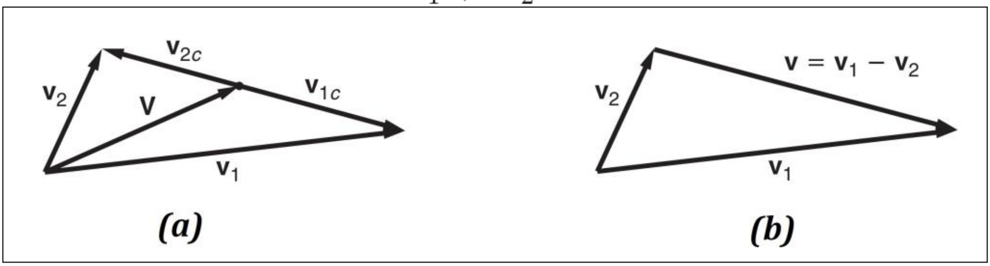

Figure2.1:

and

$$\mathbf{v}_{2c} = \mathbf{v}_2 - \mathbf{V}$$
$$= \frac{-m_1}{m_1 + m_2} (\mathbf{v}_1 - \mathbf{v}_2)$$

$\mathbf{v}_{1c}$ and $\mathbf{v}_{2c}$ lie back to back along the relative velocity vector $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ as shown in the part (*b*) of the figure.

The momenta in the *COM* system are

$$\mathbf{p}_{1c} = m_1 \mathbf{v}_{1c}$$
$$= \frac{m_1 m_2}{m_1 + m_2} (\mathbf{v}_1 - \mathbf{v}_2)$$
$$= \mu \mathbf{v}$$
$$\mathbf{p}_{2c} = m_2 \mathbf{v}_{2c}$$
$$= \frac{-m_1 m_2}{m_1 + m_2} (\mathbf{v}_1 - \mathbf{v}_2)$$
$$= -\mu \mathbf{v}$$

Here $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the system, the natural unit of mass in a two-particle system. The total momentum in the *COM* system is zero. The total momentum in the Lab frame is

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = (m_1 + m_2)\mathbf{V}$$

and since total momentum is conserved in any collision, $\mathbf{V}$ is constant. We can use this result to help visualize the velocity vectors before and after the collision.

Now look at the figure below. Part (a) visualizes the path of two colliding particles before and after the collision in the lab frame. Part (b) shows the initial velocities in the Lab and *COM* systems. All the vectors lie in the same plane, and $\mathbf{v}_{1c}$ and $\mathbf{v}_{2c}$ must be in opposite direction as the total momentum in the C system is zero. After the collision, as shown in part (c), the velocities in the *COM* system are again in opposite direction. Part (c) also shows the final velocities in the lab system. Note that the plane of part (c) is not necessarily the plane of part (a)

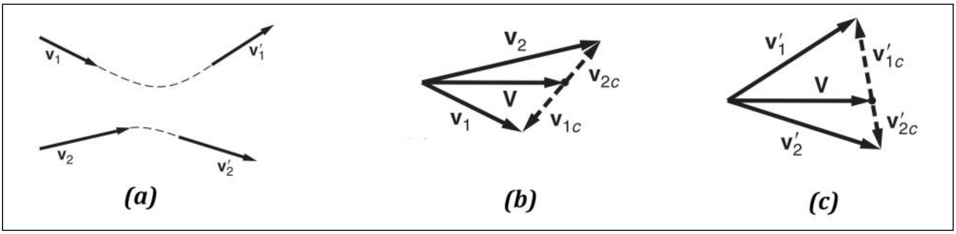

Figure 2.2:

We will derive the mathematical formulation for the elastic collision and leave the inelastic collision as the treatment will be complicated. Conservation of energy applied to the *COM* system gives, for elastic collisions,

$$\frac{1}{2}m_1v_{1c}^2 + \frac{1}{2}m_2v_{2c}^2 = \frac{1}{2}m_1v_{1c}'^2 + \frac{1}{2}m_2v_{2c}'^2$$

Total momentum is zero in the *C* system. We therefore have

$$m_1v_{1c} - m_2v_{2c} = 0 \; m_1v_{10c}$$

$$- m_2v_{20c} = 0$$

Using momentum conservation to eliminate v_{2c} and v_{2c}' from the energy equation gives

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)v_{1c}^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)v_{1c}'^2$$

or

$$v_{1c} = v_{1c}' \tag{2.1}$$

Similarly,

$$v_{2c} = v_{2c}' \tag{2.2}$$

In an elastic collision, the speed of each particle in the *COM* system is the same before and after the collision; the velocity vectors simply rotate in the scattering plane.

Now consider one of the particles, m_2 is initially at rest in the laboratory. So $\mathbf{v}_2 = 0$. This case is important as it happens in many real experiment.

In our earlier discussion of velocities before the collision, substitute $\mathbf{v}_2 = 0$.

$$\mathbf{V} = \frac{m_1}{m_1 + m_2}\mathbf{v}_1$$

$$\mathbf{v}_{1c} = \mathbf{v}_1 - \mathbf{V} = \frac{m_2}{m_1 + m_2}\mathbf{v}_1$$

$$\mathbf{v}_{2c} = -\mathbf{V} = -\frac{m_1}{m_1 + m_2}\mathbf{v}_1$$

The situation is explained in the figure below. Here the trajectories after the collision in the *COM* and Lab systems are shown. You see that $\mathbf{v}_1^0$ makes angle θ_1 , and $\mathbf{v}_2^0$ makes angle θ_2 in Lab frame. In the COM frame the velocity vectors rotates with angle Θ which is called the scattering angle.

Because these angles are in *L*, they are in principle measurable in the lab. The velocity diagrams can be used to relate θ_1 and θ_2 to the scattering angle Θ, as we shall do just now

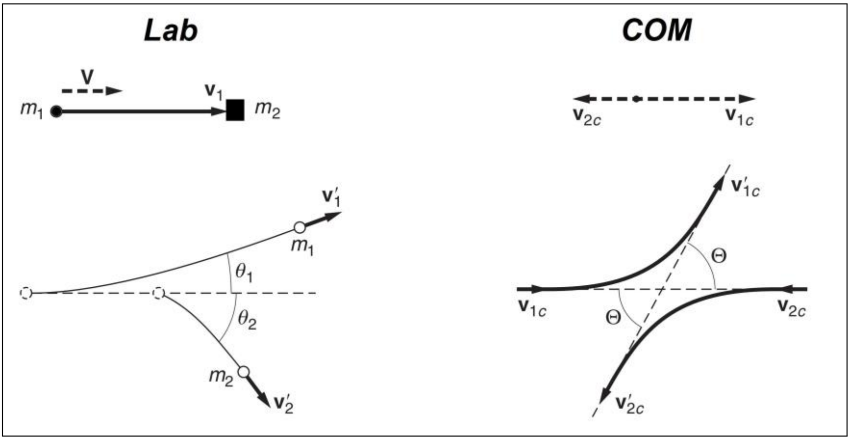

Figure2.3:

2.2 Elastic scattering when target particle at rest

Now consider the elastic scattering of a particle of mass m_1 and velocity $\mathbf{v}_1$ from a second particle of mass m_2 at rest.

The scattering angle Θ in the *COM* system is unrestricted, but the conservation laws impose limitations on the laboratory angles, as we now show. The center of mass velocity has magnitude

$$V = \frac{m_1 v_1}{m_1 + m_2} \quad (2.3)$$

The velocity of COM is parallel to $\mathbf{v}_1$. The initial velocities in the COM system are

$$_{1c} = \frac{m_2}{m_1 + m_2}\mathbf{v}_1$$

$$_{2c} = -\frac{m_1}{m_1 + m_2}\mathbf{v}_1$$

v

(2.4)

v

The mass m_1 is scattered through angle Θ in the *COM* system as shown in figure above. This essentially means that the mass m_1 deviates an angle Θ from its original direction in the COM

frame.

Now look at the diagram below. This is velocity diagram of the collision. Look at the figure 2.3 first. You must see that the initial direction of m_1 is same both in Lab and COM frame (although the magnitude of velocities are different) as well as the velocity of COM frame which is V. θ_1 is the angle between the initial and final direction of mass m_1 in the Lab frame. Θ is the angle between the initial and final direction of mass m_1 in the COM frame. As initial direction of m_1 is same both in Lab and COM frame we draw the vector direction of V, v_1, v_{1c} along same line.

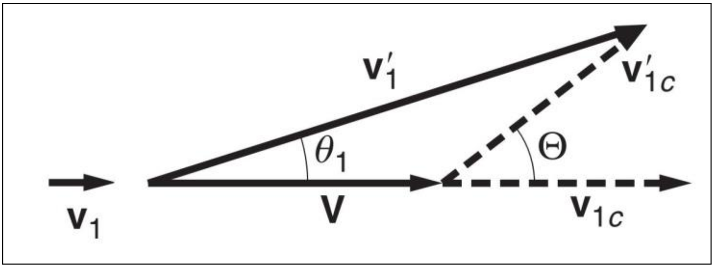

Figure 2.4:

Now it should be clear from the velocity addition rule that

$$v_1' \sin\theta = v_{1c}' \sin\Theta$$

and

$$v_1' \cos\theta = V + v_{1c}'$$

Hence we write

$$\tan\theta_1 = \frac{v_{1c}' \sin\Theta}{V + v_{1c}' \cos\Theta}$$

since the scattering is elastic, $v_{1c}' = v_{1c}$. Hence

$$\tan\theta_1 = \frac{v_{1c}\sin\Theta}{V + v_{1c}\cos\Theta} = \frac{\sin\Theta}{(V/v_{1c}) + \cos\Theta}$$

From (2.3) and 2.4 we get $V/v_{1c} = m_1/m_2$. So,

$$\tan\theta_1 = \frac{\sin\Theta}{(m_1/m_2) + \cos\Theta} \tag{2.5}$$

You can easily check few important results

$$\text{(i)} \quad \theta_1 \cong \Theta, \quad m_1 \ll m_2 \tag{2.6}$$

and

$$\text{(ii)} \quad \theta_1 = \frac{\Theta}{2}, \quad m_1 = m_2 \tag{2.7}$$

(iii) If $m_1 < m_2$, all scattering angles for m_1 in LS are possible.

(iv) If $m_1 = m_2$, scattering only in the forward hemisphere ($\theta \leq 90^\circ$) is possible.

(v) If $m_1 > m_2$, the maximum scattering angle is possible, θ_{1max}, being given by

$$\theta_{1max} = \sin^{-1}(m_2/m_1)$$

We'll prove some important relations relating the scattering angle of target mass m_2 in Lab and COM frame. Look at the figure below. The scattering angles in Lab and COM frame has been drawn together in upper left figure (a), the COM frame alone in upper right (b) and the vector diagram of final state of target mass both in Lab and COM m_2 below

(c).

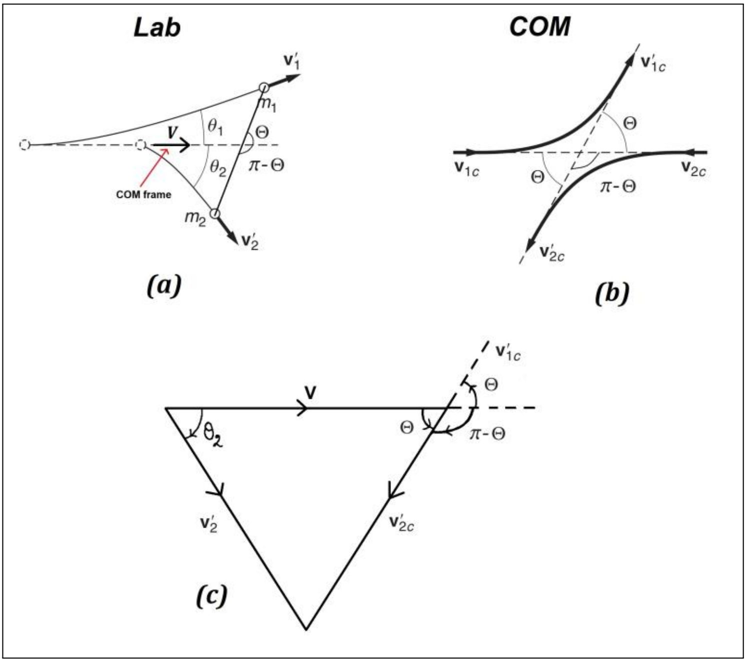

Figure 2.5:

We get by balancing the velocity vectors along *y* axis

$$v_2' \sin\theta_2 = v_{2c}' \sin\Theta$$

And by balancing the velocity vectors along *x* axis

$$v_2' \cos\theta_2 = V - v_{2c}' \cos\Theta$$

Divide first equation by second

$$\tan\theta_2 = \frac{\sin\Theta}{\frac{V}{v_{2c}'} - \cos\Theta}$$

From equations (2.1), (2.2) , (2.3) and (2.4) we easily see that the magnitude of velocity of COM frame and the velocity of m_2 after the collision in COM frame are equal, i.e

$$V = v_{2c}' \tag{2.8}$$

Hence

$$\tan\theta_2 = \frac{\sin\Theta}{1 - \cos\Theta} = \cot\frac{\Theta}{2} \tag{2.9}$$

We may write this as

$$\tan\theta_2 = \tan\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) \tag{2.10}$$

Thus

$$\theta_2 = \frac{1}{2}(\pi - \Theta) \quad \Longrightarrow \quad 2\theta_2 = (\pi - \Theta) \tag{2.11}$$

This is an important result. In the special case of $m_1 = m_2$, we already know

$$m_1 = m_2 \quad \Longrightarrow \quad \theta_1 = \frac{\Theta}{2} \quad \Longrightarrow \quad \Theta = 2\theta_2$$

Combining with (2.11) we get

$$\theta_1 + \theta_2 = \frac{pi}{2}; \quad \text{when} \quad m_1 = m_2 \tag{2.12}$$

This result we have proved before.

2.2.1 Kinetic energies in the Lab and COM frame

KE in lab frame

$$T_1 = \frac{1}{2} m_1 v_1^2$$

KE in COM frame

$$T_{1c} = \frac{1}{2}\left(m_1 v_{1c}^2 + m_2 v_{2c}^2\right)$$

Using equation (2.4) we have

$$T_{1c} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v_1^2 = \frac{m_2}{m_1+m_2}T_1$$

This result shows that the initial kinetic energy in the COM system T_c is always a fraction $m_2/(m_1 + m_2) < 1$ of the initial LAB energy. For the final COM energies, we find

$$T'_{1c} = \frac{1}{2}m_1 v'^2_{1c} = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2 v_1^2 = \left(\frac{m_2}{m_1+m_2}\right)^2 T_1$$

and

$$T'_{2c} = \frac{1}{2}m_2 v'^2_{2c} = \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2 v_1^2 = \frac{m_1 m_2}{(m_1+m_2)^2}T_1$$

The ratio of KE of m_1 before and after collision

$$\frac{T'_1}{T_1} = \frac{\frac{1}{2}m_1 v'^2_1}{\frac{1}{2}m_1 v_1^2} = \frac{v'^2_1}{v_1^2}$$

Using cosine law in figure 2.4

$$v'^2_{1c} = v'^2_1 + V^2 - 2v'_1 V\cos\theta_1$$

Hence

$$\frac{T'_1}{T_1} = \frac{v'^2_1}{v_1^2} = \frac{v_{1c}^2}{v_1^2} - \frac{V^2}{v_1^2} + 2\frac{v'_1 V}{v_1^2}\cos\theta_1$$

We already know

$$\frac{v'_{1c}}{v_1} = \frac{m_2}{m_1 + m_2} \quad \text{and} \quad \frac{V}{v_1} = \frac{m_1}{m_1 + m_2}$$

We also know $v'_{1c} \sin\Theta = v'_1 \sin\theta_1$ Using this we get

$$2\frac{v'_1 V}{v_1^2}\cos\theta_1 = 2\left(v'_{1c}\frac{\sin\Theta}{\sin\theta_1}\right)\cdot\frac{V}{v_1^2}\cos\theta_1$$

and using (2.5)

$$\frac{\sin\Theta\cos\theta_1}{\sin\theta_1} = \frac{\sin\Theta}{\tan\theta_1} = \cos\Theta + \frac{m_1}{m_2}$$

So that

$$2\frac{v'_1 V}{v_1^2}\cos\theta_1 = \frac{2m_1 m_2}{(m_1+m_2)^2}\left(\cos\Theta + \frac{m_1}{m_2}\right)$$

Doing the substitutions and some algebra (do yourself) we get

$$\frac{T'_1}{T_1} = \left(\frac{m_2}{m_1+m_2}\right)^2 - \left(\frac{m_1}{m_1+m_2}\right)^2 + \frac{2m_1 m_2}{(m_1+m_2)^2}\left(\cos\Theta + \frac{m_1}{m_2}\right)$$

This simplifies to

$$\frac{T'_1}{T_1} = 1 - \frac{2m_1 m_2}{(m_1+m_2)^2}(1 - \cos\Theta) \qquad (2.13)$$

This is the ratio of the KE of the projectile particle m_1 before and after collision. Do each step in this formulation carefully.

There can be many problems which can be formed based on this discussion. If you are thinking why did we calculate the ration of KE of the projectile particle, I would like to mention that in many experiment we have the idea about the ratio of the KE of the projectile and we need to calculate at which angle the particle detector should be placed, i.e, calculate θ_1. We will see in an example shortly.

Prove yourself that

$$\frac{T_1'}{T_1} = \frac{m_1^2}{(m_1+m_2)^2}[\cos\theta_1 \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\theta_1}]^2 \tag{2.14}$$

Example2.1: Whatisthemaximumanglethat θ_1 can attainforthecase $V>v_{1c}^{C}$?Whatis $\theta_{1\,max}$ for m_1 m_2 and $m_1 = m_2$?

Solution: The scattering angle Θ depends on the details of the interaction, but in general it can assume any value. If $m_1 < m_2$, it follows from equation (2.5) or the geometric construction in part (a) of the figure below that θ_1 is unrestricted.

However, the situation is quite different if $m_1 > m_2$. In this case θ_1 is never greater than a certain angle $\theta_{1,max}$. As figure

(b) shows, the maximum value of θ_1 occurs when $\mathbf{v}_1{}^0$ is perpendicular to $\mathbf{v}_1{}^0{}_c$.

Now look at the geometry of part (b) of the figure. You see

$$\sin\theta_{1,\max} = v_{1c}/V = m_2/m_1$$

(don't forget $v'_{1c} = v_{1c}$) If

$$m_1 \gg m_2, \theta_{1,\max} \approx m_2/m_1$$

and the maximum scattering angle in Lab frame approaches zero.

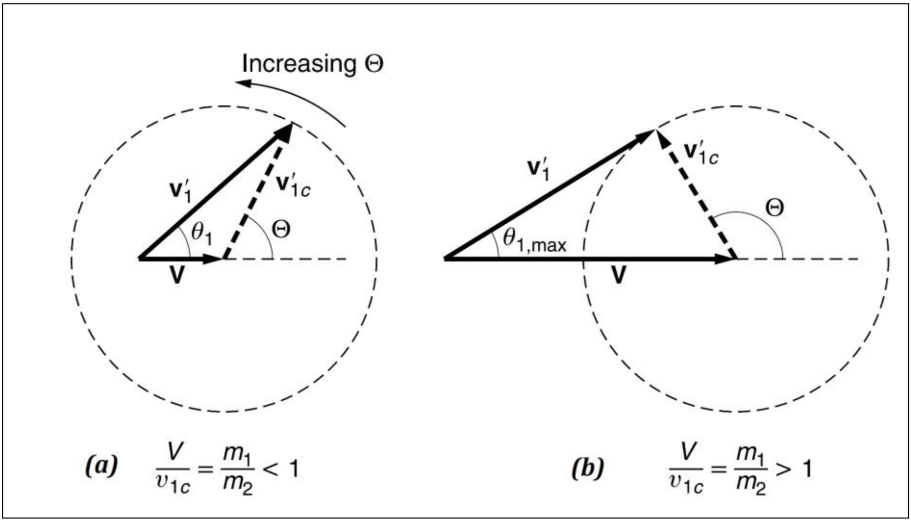

Figure 2.6:

Think on your familiar example, if $m_1/m_2 < 1$ as in (a), this is like a flow of small marbles hitting a big ball; the marbles scatter in all directions. On the other hand, if a moving big ball hits small marble, then $m_1/m_2 \gg 1$ and the big ball do not deflect much.

For $m_1 = m_2$

$$\tan\theta_1 = \frac{\sin\Theta}{1+\cos\Theta} = \tan(\Theta/2)$$

so that

$$\theta_1 = \Theta/2$$

Example 2.2: If a particle of mass m_1 collides elastically with one of mass m_2 at rest, and if m_1 mass is scattered at an angle θ_1 and the mass m_2 recoils at an angle θ_2 with respect to the line of motion of the incident particle, (see figure 2.3) then show that

$$\frac{m_1}{m_2} = \frac{\sin(2\theta_2 + \theta_1)}{\sin\theta_1}$$

Solution: This can be easily done with the help of relation between angles in COM and Lab frame. As we know from equation (2.5)

$$\tan\theta_1 = \frac{\sin\Theta}{(m_1/m_2)+\cos\Theta}$$

Again we know from (2.11)

$$\Theta = \pi - 2\theta_2$$

or $\sin\Theta = \sin(\pi - 2\theta_2) = \sin 2\theta_2$

and

$$\cos\Theta = \cos(\pi - 2\theta_2) = -\cos 2\theta_2$$

Using these

$$\tan\theta_1 = \frac{\sin\Theta}{(m_1/m_2) + \cos\Theta}$$

$$\frac{sin\theta_1}{\cos\theta_1} = \frac{\sin 2\theta_2}{(m_1/m_2) - \cos 2\theta_2}$$

Doing cross multiplication and rearrangement

$$\frac{m_1}{m_2}\sin\theta_1 = \sin\theta_1\cos 2\phi + \cos\theta_1\sin 2\theta_2 = \sin(\theta_1 + 2\theta_2)$$

hence

$$\frac{m_1}{m_2} = \frac{\sin(2\theta_2 + \theta_1)}{\sin\theta_1}$$

Example 2.3: Particles of mass m_1 elastically scatter from particles of mass m_2 at rest.

(a) At what LAB angle should a particle detector be setto detect particles that lose one-third of their momentum?

(b) Over what range m_1/m_2 is this possible?

(c) Calculate the scattering angle for $m_1/m_2 = 1$

Solution: (a) In the Lab frame given that

$$m_1 v_1' = \frac{2}{3} m_1 v_1 \quad \Longrightarrow \quad v_1' = \frac{2}{3} v_1$$

Ratio of final and initial kinetic energy of particle m_1 in Lab frame is

$$\frac{T_1'}{T_1} = \frac{\frac{1}{2} m_1 v_1'^2}{\frac{1}{2} m_1 v_1^2} = \frac{v_1'^2}{v_1^2} = \left(\frac{2}{3}\right)^2 = 1 - \frac{2 m_1 m_2}{(m_1 + m_2)^2} (1 - \cos\Theta)$$

This equation can be solved for $\cos\theta$, giving

$$\cos\Theta = 1 - \frac{5\,(m_1 + m_2)^2}{18 m_1 m_2} = 1 - y$$

we have written $y = \dfrac{5\,(m_1 + m_2)^2}{18 m_1 m_2}$

We need θ_1 here so use equation (2.5)

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + m_1/m_2} = \frac{\sqrt{2y - y^2}}{1 - y + m_1/m_2}$$

This is the angle where the detector should be placed. We know the values of m_1 and m_2 before the experiment, so the calculation of θ_1 can be done from the knowledge of the masses.

(b) Because $\tan\theta_1$ must be a real number, only values for m_1/m_2 where $2 - y \geq 0$ are possible. Therefore,

$$2-\frac{5\left(m_1+m_2\right)^2}{18 m_1 m_2} \geq 0$$

we write this as

$$-5\left(\frac{m_1}{m_2}\right)^2+26\left(\frac{m_1}{m_2}\right)-5 \geq 0$$

Now taking $x = m_1/m_2$ we write this equation as

$$-5x^2 + 26x - 5 \geq 0$$

If we make the inequality as equality, i.e, $-5x^2 + 26x - 5 = 0$,

we get $x=\frac{1}{5}$, 5 Hence

$$\frac{1}{5} \leq \frac{m_1}{m_2} \leq 5$$

(c) Substituting $m_1/m_2 = 1$ in the definition of y

$$y=\frac{5\left(m_1+m_2\right)^2}{18 m_1 m_2}=\frac{5\left(\frac{m_1}{m_2}+1\right)^2}{18 m_1 / m_2}$$
$$=\frac{5(1+1)^2}{18}=\frac{10}{9}$$

and substituting for y into $\tan\theta_1$ gives $\theta_1 = 48^\circ$