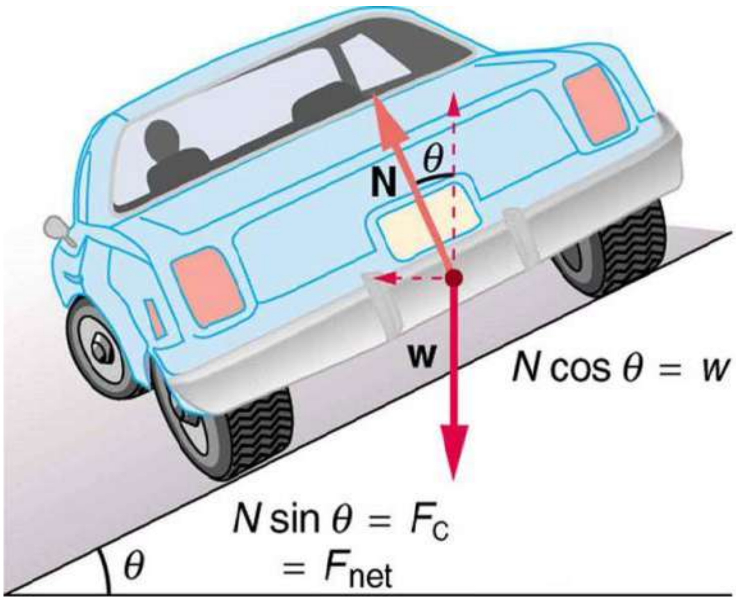
B.Sc. Mathematics (Honours)

Class Note: Tangent and Normal Component of Acceleration



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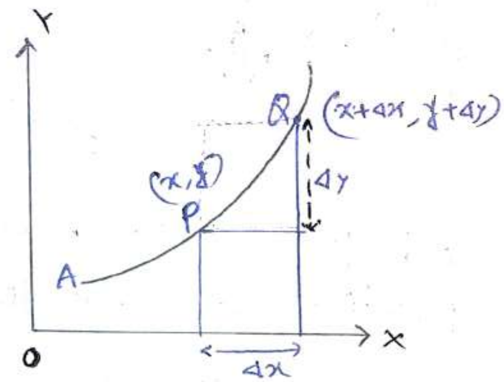
(TANGENT & NORMAL COMPONENTS OF ACCELERATION)
CONSTRAINED OF MOTION



Let a particle while moving on a plane along the path APQ be at points P and Q at time t and $t+\Delta t$ respectively. With reference to a fixed rectangular axes OX and OY, let the co-ordinates of P be (x,y) and those of Q be $(x+\Delta x, y+\Delta y)$. The displacement PQ of the particle has components Δx and Δy parallel to ox and oy respectively. Hence the velocity component parallel to OX is

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \qquad \text{—— (i)}$$

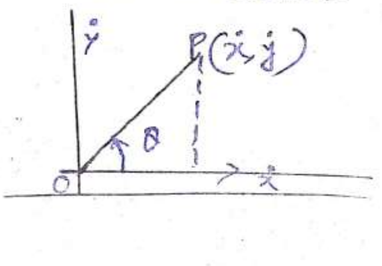
Similarly, the velocity component parallel to oy is

$$v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}.$$

<u>Note-I</u>: Resultant velocity $(v) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2+\dot{y}^2}$.

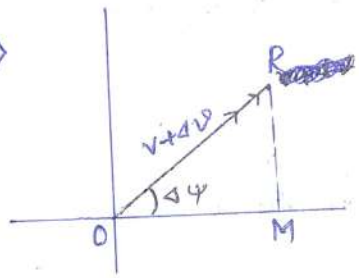
<u>Note-II</u>: Angle of resultant velocity with positive x axis

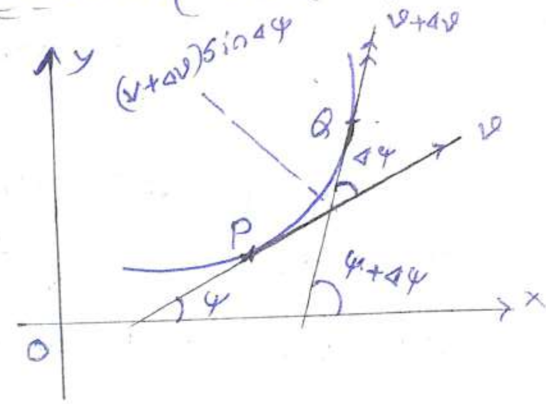
$$\theta = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$$



Tangential and Normal Accelerations :→ (Component)

Cartesian forms





Let P be the position of the particle at any time t & Q that in time $t+\Delta t$.

Let the tangent at P and Q to the curve makes angle ψ and $\psi+\Delta\psi$ respectively with x-axis.

Also let v and $v+\Delta v$ be the velocity of the particle at P and Q respectively which act along their respective angle in the direction as shown as the above figure –

Now. Tangential component of the acceleration (f_t)

$$f_t = \lim_{\Delta t \to 0} \frac{\text{Change the velocity in time } \Delta t \text{ at } P}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(\text{velocity in time } 't+\Delta t' \text{ along the tangent} - v \text{ in time } t \text{ along the tangent})}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(v+\Delta v)\cos\Delta\psi - v}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{v+\Delta v - v}{\Delta t} \qquad (\because \cos\Delta\psi \simeq 1)$$

$$= \frac{dv}{dt} \,, \quad = v \cdot \frac{dv}{ds}$$

Let f_n be the normal component of acceleration.

So,

$$f_n = \lim_{\Delta t \to 0} \frac{\text{Change of velocity in time } \Delta t \text{ along normal at } P}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\text{velocity in time } t+\Delta t \text{ along normal} - \text{velocity in time } t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(v+\Delta v)\sin\Delta\psi - 0}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(v+\Delta v).\Delta\psi}{\Delta t} \quad (\because \sin\Delta\psi = \Delta\psi)$$

$$= \lim_{\Delta t \to 0} \frac{v.\Delta\psi + \Delta v.\Delta\psi}{\Delta t}$$

$$\left(\because \Delta v.\Delta\psi \text{ is very small i.e. } \Delta v.\Delta\psi << 1\right)$$

$$= \lim_{\Delta t \to 0} v.\frac{\Delta\psi}{\Delta t}$$

$$= v\frac{d\psi}{dt} \quad \left(\frac{d\psi}{dt} = \omega \text{ is called angular velocity}\right)$$

$$= v \times \frac{d\psi}{ds}.\frac{ds}{dt}$$

$$= v^2/\rho, \quad \left(\rho = \frac{ds}{d\psi}\right)$$

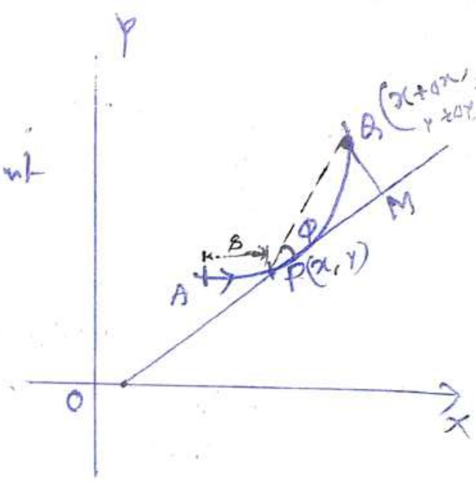
Where ρ be the radious of curvature.

Note-I. Acceleration parallel to x-axis $= \frac{d^2x}{dt^2} = \ddot{x}$

II. Accelaretion parallel to y-axis $= \frac{d^2y}{dt^2} = \ddot{y}$

III. Resultant acceleration $= \sqrt{\ddot{x}^2 + \ddot{y}^2}$.

Tangential & Normal velocity:



Tangential velocity

$$= \lim_{\Delta t \to 0} \frac{\text{Displacement along the tangent}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{PM}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\overset{\frown}{PQ} \ (\doteq \text{chord } PQ)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{PQ \cos\phi}{\Delta t} = \lim_{\Delta t \to 0} \frac{PQ}{\Delta s} \cdot \frac{\Delta s}{\Delta t} \cdot \cos\phi$$

$$= v. \quad \left(\text{When} \quad Q \to P \ni \lim_{\Delta t \to 0} \frac{PQ}{\Delta s} = 1 \ \& \ \phi = 0\right).$$

Normal velocity,

$$= \lim_{\Delta t \to 0} \frac{\text{Displacement} \perp \text{to tangent in time } \Delta t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{QM}{\Delta}$$

$$= \lim_{\Delta t \to 0} \frac{PQ \sin\phi}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{PQ}{\Delta s} \cdot \frac{\Delta s}{\Delta t} \sin\phi$$

$$= 0. \quad \left(\text{when } Q \to P \ \frac{PQ}{\Delta s} = 1 \text{ and } \phi = 0\right)$$

So, Normal velocity at P is zero.

1. A particle describes a curve $s = c\tan\psi$ with uniform speed v. Find the acceleration indicating its direction

Sol: We have

$$s = c\tan\psi \quad \text{———} \quad (1)$$

and $\dfrac{ds}{dt} = v = \text{constant}$ ($\because$ speed is uniform)

$\therefore \dfrac{d^2s}{dt^2} = 0$. So tangential component of acceleration $= 0$.

From (1) $\dfrac{ds}{d\psi} = c\sec^2\psi$

i.e. $\rho = c \cdot \sec^2\psi$

Normal component of acceleration $= \dfrac{v^2}{\rho} = \dfrac{v^2}{c} \cdot \cos^2\psi$

Hence the acceleration is $\dfrac{v^2}{c}\cos^2\psi$ whose direction is along the normal to the curve.

2. A particle moves in a plane in such a manner that its tangential and normal accelerations are equal and its velocity varies as $e^{\tan^{-1}\frac{s}{c}}$, s being the length of the arc of the curve measured from a fixed point. Find the path.

Hints: $v\dfrac{dv}{ds} = v^2/\rho = v^2 \cdot \dfrac{d\psi}{ds}$ $\quad \left(\because \rho = \dfrac{ds}{d\psi}\right)$

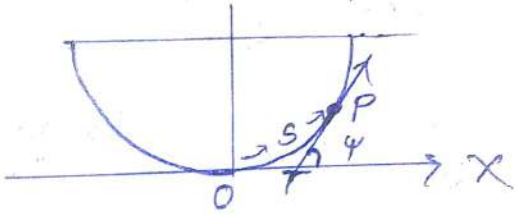
$\Rightarrow v = ce^{\psi}$

Again $v \propto e^{\tan^{-1}\frac{s}{c}}$.

$s = c\tan(\psi + \epsilon)$ is the required path of the particle.

3. A point moves along the arc of a cycloid in such a manner that the direction of motion rotates with constant angular velocity; show that the acceleration of the moving point is constant in magnitude.

Hints:



Let O be the vertex of the cycloid and P be the position of the point at any time t. Let the tangent at P make an angle ψ with the tangent OX at O. Let arc OP = s. Then the intrinsic equation (equation involving s and ψ) of the cycloid is

$$s = \cancel{4\sin\psi}\ 4a\sin\psi$$

Since the direction of motion (i.e. TP) rotates with constant angular velocity;

$$\therefore \frac{d\psi}{dt} = \text{constant} = \omega \text{ (say)}$$

$\therefore$ Tangential acceleration $= \dfrac{d^2s}{dt^2} = -4a\omega^2\sin\psi$

Normal acceleration $= v^2/\rho = 4a\omega^2\cos\psi$.

4. A particle moves in a plane, its velocity parallel to the axis of x and y being $u+ey$ and $v+ex$ respectively. Find the equation of the path

Hints: $\frac{dx}{dt} = u + ey$, $\frac{dy}{dt} = v + ex$

$$\therefore \frac{dy}{dx} = \frac{v + ex}{u + ey}$$

$$\Rightarrow (u + ey)\,dy = (v + ex)\,dx \;\ldots\ldots$$

5. A particle describe a plane curve such that the tangential and normal acceleration are each constant throughout the motion, prove that the angle ψ through which the direction of motion turns in time t, is of the form, $\psi = A \log(1 + Bt)$, where A and B are constant.

Hints: $\frac{d^2 s}{dt^2} = \lambda_1$, $\frac{v^2}{\rho} = \frac{\left(\frac{ds}{dt}\right)^2}{\left(\frac{ds}{d\psi}\right)} = \lambda_2$ (λ_1, λ_2 are const)

$$\Rightarrow \frac{ds}{dt} = \lambda_1 t + c \quad \text{—(I)}, \qquad \frac{ds}{dt} \cdot \frac{ds}{dt} \cdot \frac{d\psi}{ds} = \lambda_2$$

$$\Rightarrow \frac{ds}{dt} \cdot \frac{d\psi}{dt} = \lambda_2 \quad \text{—(II)}$$

$$\therefore (\lambda_1 t + c)\frac{d\psi}{dt} = \lambda_2$$

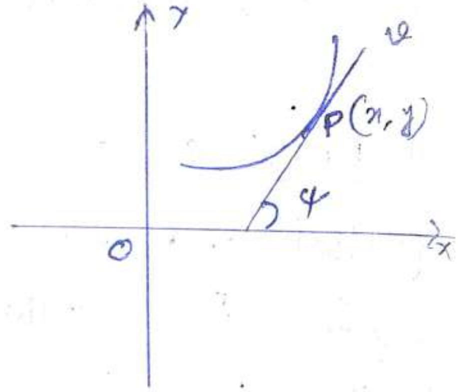
$$\Rightarrow d\psi = \lambda_2 \cdot \frac{1}{\lambda_1 t + c}\,dt$$

$$\psi = \frac{\lambda_2}{\lambda_1} \log(\lambda_1 t + c) - \frac{\lambda_2}{\lambda_1}\log c_1 \quad \left[\text{Let } t = 0, \psi = 0 \;\; \therefore c_1 = -\frac{\lambda_2}{\lambda_1}\log c\right]$$

$$= \frac{\lambda_2}{\lambda_1} \log\left(\frac{\lambda_1}{c_1} t + \frac{c}{c_1}\right)$$

$$= A \log(1 + Bt).$$

6. A particle moves in the curve $y = a\log_e \sec(x/a)$ in such a way that the tangent to the curve rotates uniformly. prove that the resultant acceleration of the particle varies as the square of the radious of curvature.



Hints: $\tan\psi = \frac{dy}{dx} = \tan x/a$

$\therefore \psi = x/a$

$\Rightarrow x = a\psi$

again given that $\frac{d\psi}{dt} = \omega$

If v be the velocity at P, then

$$v = \frac{ds}{dt} = \frac{ds}{dx}\cdot\frac{dx}{d\psi}\cdot\frac{d\psi}{dt} = \frac{a}{\cos\psi}\cdot\omega \quad \left(\because \frac{ds}{dx} = \frac{1}{\cos\psi}\right)$$

$$\therefore f_T = \frac{dv}{dt}$$

$$= \frac{d}{d\psi}(a\omega\sec\psi)\cdot\frac{d\psi}{dt}$$

$$= a\omega^2\sec\psi\cdot\tan\psi$$

$$\boxed{\begin{aligned} &\tan\psi = \frac{dy}{dx} \\ &\Rightarrow \frac{\sin\psi}{\frac{dy}{ds}} = \frac{\cos\psi}{\frac{dx}{ds}} \\ &\text{i.e., } \frac{dx/ds}{\cos\psi} = \frac{dy/ds}{\sin\psi} \end{aligned}}$$

$$f_N = v^2/\rho$$

$$= \frac{a^2\omega^2\sec^2\psi}{a\sec\psi} \quad \left(\because \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = a\sec\psi\right)$$

$$= a\omega^2\sec\psi$$

$$\therefore f = \sqrt{f_T^2 + f_N^2}$$

$$= \frac{\omega^2}{a}(a\sec\psi)^2 = \frac{\omega^2}{a}\cdot\rho^2 \Rightarrow \boxed{f \propto \rho^2}$$

7/ A particle describes the curve whose intrinsic equ^n is $s = c \tan\psi$ with uniform speed v. Show that the normal component of acceleration at the point $(c, \pi/4)$ is $v^2/2c$

Hints: $\therefore f_N = \frac{v^2}{\rho}$

again, $\rho = \frac{ds}{d\psi} = \frac{d}{d\psi}(c\tan\psi) = c\sec^2\psi$

$\rho\Big]_{at\ \psi=\pi/4} = 2c$. So, $f_N = \frac{v^2}{\rho} = v^2/2c$ (proved)

8/ A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $\frac{v^2c}{s^2+c^2}$, prove that curve is a catenary.

Hints: $v\frac{dv}{ds} = \frac{v^2c}{s^2+c^2} \Rightarrow \frac{dv}{v} = \frac{c\,ds}{s^2+c^2} \Rightarrow \log v = \tan^{-1}(s/c)$

since speed is uniform, we take $v = k$ and $\log k = \psi$ which implies that $s = c\tan\psi$. (is a catenary)

9/ If the tangential and normal acceleration of a particle describe a plane curve be constant through its motition Find the radious of curvature.

Hints: $\frac{d^2s}{dt^2} = c_1$, $\frac{v^2}{\rho} = c_2$

$\Rightarrow d\left(\frac{ds}{dt}\right) = c_1\,dt$

$\Rightarrow v = c_1t + c_3$

$\therefore \rho = \frac{v^2}{c_2}$

$= \frac{(c_1t + c_3)^2}{c_2} = (At+B)^2$

$\boxed{\therefore \rho = (At+B)^2}$ $A = \frac{c_1}{\sqrt{c_2}}$, $B = \frac{c_3}{\sqrt{c_2}}$