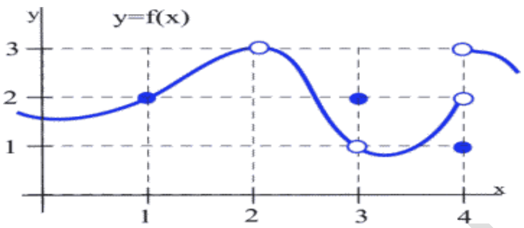
B.Sc. Mathematics (Honours/Major)

Class Note: Limit and Continuity



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Introduction to Limits:

The concept of limits plays a fundamental role in calculus and real analysis. It allows us to describe the behaviour of a function as its input approaches a particular value. Limits are essential in understanding continuity, derivatives, and integrals.

The Intuition Behind Limits:

Imagine a function $f(x)$ representing the position of an object at time x. The limit of f(x) as x approaches a specific value c can be interpreted as the position the object would reach if it could be infinitely close to c, without actually reaching c.

Limits Using the ε- δ Approach:

The ε - δ approach is a rigorous method to define limits precisely. It states that a function $f(x)$ approaches a limit L as x approaches a value c if for every positive ε (epsilon), there exists a positive δ (delta) such that |f(x) - L| < ε whenever 0 < |x - c| < δ.

Example 1: Find the limit of f(x) = 3x + 2 as x approaches.

Solution: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3}$. Then, for all $0 < |x - 4| < \delta$

$$|f(x) - 14| = |3x + 2 - 14| = |3(x - 4)| = 3|x - 4| < 3.\frac{\varepsilon}{3} = \varepsilon.$$

Sequential Criterion for Limits:

The sequential criterion for limits is an alternative way to define limits. According to this criterion, for a function $f(x)$ to have a limit L as x approaches c, every sequence $\{x_n\}$ that converges to c must also cause the sequence $\{f(x_n)\}$ to converge to L.

Example 2: Prove that the limit of $f(x) = x^2$ as x approaches 3 is 9 using the sequential criterion.

Solution: Let $\{x_n\}$ be any sequence that converges to 3. As $\{x_n\}$ converges to 3, we have $lim(x_n) = 3$. Now, $\lim (f(x_n)) = \lim (x_n^2) = 3^2 = 9$.

Divergence Criteria:

Divergence criteria are used to determine if a function approaches infinity or negative infinity as the input approaches a particular value.

Example 3: Determine the divergence of the function $g(x) = \frac{1}{x^2}$ as x approaches 0.

Solution: As x approaches 0, $g(x) = \frac{1}{x^2}$ becomes larger and larger without bound, indicating divergence to positive infinity.

Limit Theorems:

Sum and difference:

- $lim\,(f(x) + g(x)) = lim\,f(x) + lim\,g(x)$

- $lim\ (f(x)\ -\ g(x))\ =\ lim\ f(x)\ -\ lim\ g(x).$

Product:

$lim\ (f(x)\ *\ g(x))\ =\ lim\ f(x)\ *\ lim\ g(x)$ (if at least one limit is non-zero).

Quotient:

$$lim\ (f(x)\ /\ g(x))\ =\ lim\ f(x)\ /\ lim\ g(x)\ (if\ lim\ g(x)\ \neq\ 0).$$

Composition of limits: This theorem states that if a function h(x) depends on another function g(x), then the limit of h(x) as x approaches c can be found by first taking the limit of h(g(x)) as g(x) approaches g(c). This means:

$lim\ h(g(x))\ as\ x -> c\ =\ lim\ h(t)\ as\ t -> g(c)$, where $t\ =\ g(x)$.

Example 4: Evaluate the limit of $\boldsymbol{h(x)} = \frac{\boldsymbol{3x^2-5}}{\boldsymbol{2x+1}}$ as x approaches-1.

Solution: Since both the numerator and denominator of $h(x)$ are continuous at $x\ =\ -1$, we can use the limit theorems to evaluate the limit: $\lim_{x\to-1}\left(h(x)\right) = \lim_{x\to-1}\left(\frac{3x^2-5}{2x+1}\right) = \frac{3(-1)^2-5}{2(-1)+1} = \frac{3-5}{-1} = -2.$

One-sided Limits:

Left-hand limit: A function f(x) has a left-hand limit L at x = c if for any positive epsilon ε, there exists a positive delta δ such that for all x in the domain of f satisfying $c\ -\ \delta\ <\ x\ <\ c$, we have $|f(x)\ -\ L|\ <\ \varepsilon$. Symbolically, $\lim_{x\to c-} f(x) = L.$

Right-hand limit: Similarly, a function f(x) has a right-hand limit L at x = c if for any positive epsilon ε, there exists a positive delta δ such that for all x in the domain of f satisfying $c\ <\ x\ <\ c\ +\ \delta$, we have $|f(x)\ -\ L|\ <\ \varepsilon$. Symbolically, $\lim_{x\to c+} f(x) = L.$

Infinite Limits and Limits at Infinity:

Infinite Limits:

A function f(x) has an infinite limit of positive infinity ($+\infty$) as x approaches a (denoted $\lim_{x\to a} f(x) = +\infty$) if, for any positive value M, there exists a corresponding δ such that for all x in the domain of f satisfying

$$|x\ -\ a|\ <\ \delta, f(x)\ >\ M.$$

Similarly, f(x) has an infinite limit of negative infinity ($-\infty$) as x approaches a (denoted denoted $\lim_{x\to a} f(x) =\ -\infty$) if, for any negative value N, there exists a corresponding δ such that for all x in the domain of f satisfying

$$|x\ -\ a|\ <\ \delta, f(x)\ <\ N.$$

Examples:

- $f(x)\ =\ 1/x$ as x approaches 0 has an infinite limit of positive infinity.
- $f(x)\ =\ x^2$ as x approaches negative infinity has an infinite limit of positive infinity.

Limits at Infinity:

While infinite limits deal with specific input values, "limits at infinity" focus on the function's behaviour as its input approaches positive or negative infinity.

Formally: A function f(x) has a limit of L as x approaches positive infinity (+∞) (denoted $\lim_{x\to\infty+} f(x) = L$) if, for any positive ε, there exists a corresponding M such that for all $x > M, |f(x) - L| < \varepsilon$.

Similarly, f(x) has a limit of L as x approaches negative infinity $(-\infty)$ (denoted $\lim_{x\to\infty-} f(x) = L$) if, for any positive ε, there exists a corresponding N such that for all $x < N, |f(x) - L| < \varepsilon$.

Examples:

- $f(x) = 1/x$ as x approaches positive infinity has a limit of 0.
- $f(x) = x^2$ as x approaches negative infinity has no limit (oscillates between positive and negative values).

Continuity:

In calculus, continuity is a fundamental concept that ensures functions behave smoothly, without abrupt jumps or holes.

Formal Definition of Continuity:

A function f(x) is considered continuous at a point c in its domain if, for any given positive ε (epsilon), there exists a positive δ (delta) such that:

For all x in the domain of f satisfying $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$.

Intuitively, this means: no matter how "close" we get to the point c (within a δ distance), the function's output f(x) stays "close" to its value at that point f(c) (within an ε distance).

Examples:

- The constant function $f(x) = 2$ is continuous everywhere because for any chosen ε, we can always find a δ sufficiently small to ensure all points within that distance of x are mapped to values within ε of 2.
- The function $f(x) = 1/x$ is not continuous at $x = 0$. Because as x approaches 0, the value of f(x) = 1/x grows infinitely large, violating the bound set by ε no matter how small we choose δ.

Algebra of Continuous Functions:

Addition/Subtraction: If f and g are continuous at $x = a$, then both $f(x) + g(x)$ and $f(x) - g(x)$ are continuous at $x = a$.

Multiplication: If f and g are continuous at x = a, then $f(x) * g(x)$ is continuous at $x = a$, provided at least one of them is non-zero at that point.

Division: If f and g are continuous at $x = a$, and $g(a) \neq 0$, then $f(x) / g(x)$ is continuous at $x = a$.

Composition: This theorem states that if f and g are continuous functions, and $g(a) = c$, then the composite function $h(x) = f(g(x))$ is continuous at $x = a$.

Sequential Criterion for Continuity:

The sequential criterion for continuity states that a function is continuous at a point if and only if the function preserves the convergence of every sequence that approaches that point.

Formal Statement:

Let f be a function defined on a subset of the real numbers and let c be a point in that subset. Then, f is continuous at c if and only if for every sequence $\{x_n\}$ in the domain of f that converges to c (that is, $\lim_{n\to\infty} x_n = c$), the sequence $(f(x_n))$ converges to f(c) (that is, $\lim_{n\to\infty} f(x_n) = f(c)$).

Example: Determine if the function $\boldsymbol{f(x) = 1/x}$ is continuous at $\boldsymbol{x = 2}$.

Solution: Let $\{x_n\}$ be any sequence that converges to 2. As $\{x_n\}$ converges to 2, we have $lim(x_n) = 2$. Now, $lim(f(x_n)) = lim(1/x_n) = 1/lim(x_n) = 1/2$. Since $lim(f(x_n))$ is not equal to $f(2) = 1/2$, the function is not continuous at $x = 2$, and it has a discontinuity at that point.

Continuous Functions on an Interval: A function is continuous on an interval if it is continuous at every point within that interval.

Examples:

- The function $f(x) = x^2$ is continuous on the open interval $(-\infty, \infty)$.
- The function $f(x) = 1/x$ is not continuous on any interval containing 0, as it becomes undefined at that point.
- The function $f(x) = |x|$ is continuous on the closed interval $[-\infty, \infty]$.

Conclusion:

Limits and continuity are fundamental concepts in calculus and real analysis. The epsilon-delta approach and the sequential criterion provide a rigorous foundation for precise limit evaluation. Limit theorems empower us to tackle complex limits, and one-sided and infinite limits enable the analysis of function behaviour from specific perspectives. Continuity ensures smooth function behaviour. Through practice with diverse examples, students can strengthen their understanding of these crucial mathematical concepts.

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