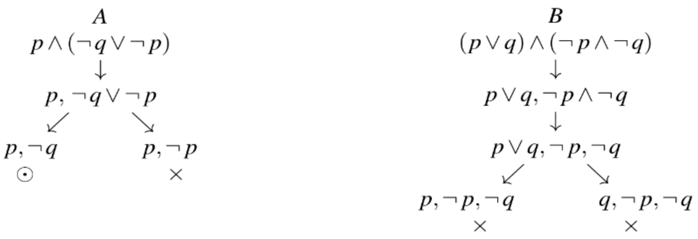
B.Sc. Mathematics (Honours/Major)

Class Note: Introduction to Logic



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1. Introduction:

Logic is the foundation of mathematics and plays a crucial role in developing sound reasoning and proof techniques. This class dives into the first steps of logical reasoning, focusing on propositional logic.

2. Propositions:

A proposition is a declarative statement that is either true or false, but not both at the same time.

We denote propositions by letters like p, q, r, etc.

Examples:

- "2 + 2 = 4",
- "The Earth is round".

3. Truth Tables:

Truth tables display the truth values of compound propositions based on the truth values of their component propositions.

We use symbols like T (true) and F (false) to represent truth values.

4. Basic Logical Operators:

a. Negation:

NOT operator (~ or ¬) flips the truth value of a proposition.

Truth table:

p	~p
T	F
F	T

b. Conjunction:

Conjunction, often represented by the symbol "∧" or simply the word "and," is a fundamental logical operator that connects propositions.

Truth table:

p	q	p ∧ q
T	T	T
T	F	F

c. Disjunction:

Disjunction, often symbolized by "∨" or the word "or," is another fundamental logical operator that joins propositions.

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A disjunction of two propositions p and q is true if at least one of p or q is true, or even if both are true. It is only false when both p and q are false.

Truth table:

p	q	p ∨ q
T	T	T
T	F	T

5. Implications and Biconditionals:

a. Implication (Conditional):

Implication, often symbolized by "→" or the phrase "if-then," is a cornerstone of logical reasoning. It connects propositions, expressing a relationship between a hypothesis and a conclusion.

The statement "If p, then q" ($p \rightarrow q$) is true only when:

- p is false (the hypothesis is not true).
- q is true (the conclusion is true).

Truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T

b. Biconditional:

The biconditional, often symbolized by "↔" or the phrase "if and only if," takes the "if-then" relationship of implication a step further. It demands a deeper connection, a perfect mirror image between two propositions.

The statement "p if and only if q" ($p \leftrightarrow q$) is true only when both p and q have the same truth value.

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Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F

6. Converse, Contrapositive, and Inverse:

a. Converse: The converse flips the positions of p and q, resulting in the statement "if q, then p." The converse is not logically equivalent to the original implication.

   Example: "If it rains, the ground is wet." (Original implication)

   Converse: "If the ground is wet, it rains." (Not always true! The ground could be wet due to other reasons)

b. Contrapositive: The contrapositive negates both p and q, resulting in the statement "if not q, then not p."

   The contrapositive is logically equivalent to the original implication. This means if the original implication is true, the contrapositive is also true, and vice versa.

   Example: "If it rains, the ground is wet." (Original implication)

   Contrapositive: "If the ground is not wet, then it did not rain." (Logically equivalent, true if the original statement is true)

c. Inverse: The inverse negates only p, resulting in the statement "if not p, then not q."

   The inverse is not logically equivalent to the original implication. Just because p is not true doesn't necessarily mean q is not true either.

   Example: "If it rains, the ground is wet." (Original implication)

   Inverse: "If it doesn't rain, then the ground is not wet." (Not always true! The ground could be dry due to other reasons)

7. Precedence of Logical Operators:

Negation has the highest precedence, followed by conjunction, then disjunction, implication, and finally biconditional. Use parentheses to clarify order of operations when needed.

8. Propositional Equivalence:

Two propositions, p and q, are said to be logically equivalent if they have the same truth value for every possible assignment of truth values to their variables.

To prove or disprove propositional equivalence, we rely on two main tools:

- Truth tables: These systematically list all possible combinations of truth values for the variables and evaluate the resulting truth values of the propositions. If the tables match, the propositions are equivalent.
- Logical equivalences: These are established rules that allow us to manipulate and simplify propositions while preserving their truth value.
  Example: p ∧ q is the same as q ∧ p (order doesn't matter for conjunction)

9. Predicates and Quantifiers:

a. Predicates:

Predicates are statements about objects, not just facts. They involve variables representing these objects and express properties or relationships.

Examples:

- "x is even."
- "y is greater than 5."
- "z loves chocolate."

Unlike propositions, predicates are not inherently true or false. They become meaningful when assigned specific values to their variables.

b. Quantifiers:

Quantifiers are symbols used to express how many objects in a domain satisfy a certain property within a predicate. They tell us if the statement applies to "all," "some," or "none" of the objects.

Types of Quantifiers:

i. Universal quantifier (∀): "For all x..." This means the statement holds true for every single object in the domain.

   Example: "$\forall x, x\text{^}2 \geq 0$" (for all real numbers x, x squared is greater than or equal to zero).

ii. Existential quantifier (∃): "There exists an x such that..." This means the statement holds true for at least one object in the domain.

   Example: "$\exists y, y$ is a prime number greater than 10" (there exists a prime number greater than 10).

10. Binding Variables and Negations:

Binding Variables:

Imagine a variable as a placeholder, representing any object within a specific domain. Quantifiers like "∀" (for all) and "∃" (there exists) bind these variables, defining their range.

Examples:

- "$\forall x, x^2 > 0$": This binds x to all real numbers. Every real number's square is greater than 0 (except 0 itself).
- "∃y, y is prime and even. This binds y to 2 as no even number is prime except 2.

Negation:

Negation (symbolized by "∼" or "¬") flips the truth value of a statement. "True" becomes "false," and vice versa. But be cautious with quantifiers! Negating a quantifier changes its meaning.

a. Negation and Universal Quantifier:

Negating "$\forall x, P(x)$" becomes "$\exists x, \sim P(x)$": This means there exists at least one object that doesn't satisfy property P.

Example: "¬($\forall x, x$ is even)" becomes "$\exists x, x$ is not even" (there exists at least one odd number).

b. Negation and Existential Quantifier:

Negating "$\exists x, P(x)$" becomes "$\forall x, \sim P(x)$": This means for all objects; property P doesn't hold.

Example: "$\neg(\exists y, y^2 = -1)$" becomes "$\forall y, y^2 \neq -1$" (no real number squared equals -1).

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