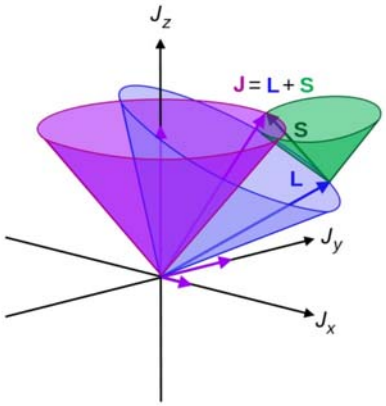
Angular Momentum

In quantum mechanics, the angular momentum operator is one of several related operators analogous to classical angular momentum. The angular momentum operator plays a central role in the theory of atomic physics and other quantum problems involving rotational symmetry. In both classical and quantum mechanical systems, angular momentum (together with linear momentum and energy) is one of the three fundamental properties of motion.

There are several angular momentum operators: **total angular momentum (** usually denoted **J**), **orbital angular momentum (** usually denoted **L) and spin angular momentum (** spin for short, usually denoted **S) .** The term angular momentum operator can (confusingly) refer to either the total or the orbital angular momentum. Total angular momentum is always conserved.



"Vector cones" of total angular momentum **J** (purple), orbital **L** (blue), and spin **S** (green). The cones arise due to quantum uncertainty between measuring angular momentum components

Fig1

1.1 Simultaneous specification of several Properties

It is postulated that if the state function ψ is an eigen function of the operator Â with eigen values s, then a measurement of the physical property A is certain to yield the result s. If ψ is simultaneously an eigen function of the two operator Â and $\hat{B}$, that is , if Âψ=sψ and $\hat{B}$ψ=tψ, then we can simultaneous assign definite values to the physical quantities A and B. When will it be possible for ψ to be simultaneously an eigenfunction of two different operators ? First, a necessary condition for the existence of a complete set of simultaneous eigenfunction of the two operators is that the operators commuting with each other. Conversely, if Â and $\hat{B}$, are two commuting operators that correspond to physical quantities , then there exists a complete set of functions that are eigenfunctions of both Â and $\hat{B}$. Thus , if [Â , $\hat{B}$]=0, then ψ can be an eigen function of both Â and $\hat{B}$.

Angular Momentum

We know from our previous discussion that the commutator $\hat{A}$ and $\hat{B}$, is defined as

$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

The following commutator identities are helpful in evaluating commutators:

$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ ……..(1)

$[\hat{A}, \hat{A}^n] = 0$, n=1,2,3… ……(2)

$[k\hat{A}, \hat{B}] = [\hat{A}, k\hat{B}] = k[\hat{A}, \hat{B}]$ …….(3)

$[\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$, $[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$ …….. (4)

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}\hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$, $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$ ……(5)

Where k is constant and the operator are assumed to be linear.

Example:

1. $[\partial/\partial x, x] = 1$

Proof: $[\partial/\partial x, x]f(x) = \partial/\partial x\{x.f(x)\} - x\partial/\partial x\{f(x)\}$

$= xf'(x) + f(x) - xf'(x) = 1.f(x)$

So, $[\partial/\partial x, x]f(x) = 1.f(x)$

Or, $[\partial/\partial x, x] = 1$ (proved) …….. ……(6)

2.

$$[\hat{x}, \hat{p}_x] = i\hbar$$

Proof: $[\hat{x}, \hat{p}_x] = [x, \frac{\hbar}{i}\partial/\partial x] = \frac{\hbar}{i}[x, \partial/\partial x]$

$$= -\frac{\hbar}{i}[\partial/\partial x, x] = -\frac{\hbar}{i}$$

$$[\hat{x}, \hat{p}_x] = i\hbar$$

…… (7)

3.

$$[\hat{x}, \hat{p}_x^2] = 2i\hbar^2 \partial/\partial x$$

Proof: $[\hat{x}, \hat{p}_x^2] = [\hat{x}, \hat{p}_x]\hat{p}_x + \hat{p}_x[\hat{x}, \hat{p}_x]$

$$= i\hbar\frac{\hbar}{i}\partial/\partial x + \frac{\hbar}{i}\partial/\partial x\, i\hbar$$

$$= 2i\hbar^2 \partial/\partial x$$

……(8)

4.

$$[\hat{x}, \hat{H}] = [\hat{x}, \hat{T}+\hat{V}] = [\hat{x}, \hat{T}] + [\hat{x}, \hat{V}(x,y,z)] = [\hat{x}, \hat{T}]$$

$$= [x, (1/2m)(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)]$$

$$= (1/2m)[\hat{x}, \hat{p}_x^2] + (1/2m)[\hat{x}, \hat{p}_y^2] + (1/2m)[\hat{x}, \hat{p}_z^2]$$

$$= (1/2m).\, 2i\hbar^2 \partial/\partial x + 0 + 0$$

$$[\hat{x}, \hat{H}] = \frac{i\hbar^2}{2m}\partial/\partial x = \frac{i\hbar}{m}\hat{p}_x \quad \text{(Proved)}$$

…… (9)

The above commutators have important physical consequences . Since $[\hat{x}, \hat{p}_x] \neq 0$, we cannot expect the state function to be simultaneously an eigenfunction of $\hat{x}$ and of $\hat{p}_x$. Hence we cannot simultaneously assign definite values to x and p_x , in agreement with the uncertinity principle. Since $\hat{x}$ and $\hat{H}$ do not commute, we cannot expect to assign definite values to the energy and x coordinate at the same time . A stationary state (which has a definite energy) shows a spread of possible values for x, the probabilities for observing various values of x being given by Born postulate.

For a state function ψ that is not an eigen function of of Â, we get various possible outcomes when we measure A in identical system. We want some measure of the of the spread or dispersion in the set of observed values A_i. If <A> is the average of these values, then the deviation of each measurement from the average is A_i-<A>. If we averaged all the deviations , we would get zero, since positive and negative deviations would cancel. Hence to make all

deviations positive, we square them. The average of the squares of the deviations is called the variance of A , symbolized in statistics by σ^2_A and in quantum mechanics by

$(\Delta A)^2 \equiv \sigma^2_A \equiv \langle (A-\langle A\rangle)^2\rangle = \int\psi^*(A-\langle A\rangle)^2\psi d\tau$ ………………………………… (10)

Where the average –value expression was used. The definition (10) is equivalent to

$(\Delta A)^2 = \langle A^2 \rangle - \langle A\rangle^2$ ……………………………………………(11)

The positive square root of the variance is called the standard deviation, σ_A or ΔA. The standard deviation is the most commonly used measure of spread, and we shall take it as the measure of the 'uncertainty' in the property A.

For the product of the standard deviations of two properties of a quantum mechanical syatem whose state function is ψ, one can show that

$(\Delta A)\ (\Delta B) \geq 1/2\ \left|\int\psi^*[\hat{A},\hat{B}]\psi d\tau\right|$ ………………………………….. (12)

If $\hat{A}$ and $\hat{B}$ commute, then the integral in equation (12) is zero, and ΔA and ΔB may both be zero, in agreement with the previous discussion.

As an example of equation (12), we find, using equation (7) and $|z_1 z_2| = |z_1|\,|z_2|$

$$\Delta x \Delta p_x \geq \frac{1}{2}\left|\int\psi^*[\hat{x},\hat{p}_x]\psi d\tau\right|$$

$$= \frac{1}{2}\left|\int\psi^* i\hbar\psi d\tau\right|$$

$$= \frac{1}{2}\hbar|i|\left|\int\psi^*\psi d\tau\right|$$

$$\Delta x \Delta p_x \geq \frac{1}{2}\hbar$$

……………………………… (13)

Equation (13) is the quantitative statement of the **Heisenberg uncertainty principle**.

Now consider the possibility of simultaneously assigning definite values to three physical quantities: A,B and C, suppose

$[\hat{A},\hat{B}]=0$ ………………………….. (14)

$[\hat{A},\hat{C}] =0$ …………………………….. (15)

Is this enough to ensure that there exist simultaneous eigenfunctions of all three operator? Equation (14) ensures that we can construct a common set of ensures that we can construct a common set of eigenfunctions for $\hat{A}$ and $\hat{B}$;Eq (15) ensures that we can construct a common set of eigenfunctions for $\hat{A}$ and $\hat{C}$. If these two sets of eigenfunctions are the same, then we will have a common set of eigenfunctions for all three operators. Hence we ask: Is the set of eigen functions of linear operator $\hat{A}$ uniquely determined? The answer is, in general ,no. If there is more than one independent eigen function corresponding to an eigen value of $\hat{A}$ (that is degeneracy), then any linear combination of the eigen functions of the degenerate eigenvalue is an eigenfunction of $\hat{A}$. It might well be that the proper linear combination needed to give eigenfunctions of $\hat{B}$ would differ from the linear combinations that give eigenfunctions of $\hat{C}$ .It turns out that, to have a common complete set of eigenfunctions of all three operators, we require that $[\hat{B},\hat{C}]$=0, in addition to equation (14) &(15). ***To have a complete set of functions that are simultaneous eigenfunctions of several operators, each operator must commute with every other operator.***

1.2 Vectors

Physical properties (for example, mass, length, energy) that are completely specified by their magnitude are called **scalars**. Physical properties **(** for example, force, velocity, momentum) that require specification of both magnitude and direction are called **vectors.** A vector is represented by a directed line segment whose length and direction give the magnitude and direction of the property.

The sum of two vectors A and B is defined by the following procedure: slide the first vector so that its tail touches the head of the second vector, keeping the direction of the first vector fixed. Then draw a new vector from the tail of the second vector to the head of the first vector. See figure 2 . The product of a vector and a scalar, cA , is defined as a vector of length $|c|$ times the length of **A** with the same direction as A if c is positive, or the opposite direction to A if c is negative.

To obtain an algebraic (as well as geometric) way of representing vectors, we set up Cartesian coordinate in space. We draw a vector of unit length directed along the positive x axis and call it **i. (** No connection with i=√-1). Unit vectors in the positive y and z directions are called j and k (figure 3). To represent any vector A in terms of the three unit vectors, we first slide A so that its tail is at the origin, preserving its direction during this process. We then find the projections of A on the x,y,z axies: A_x, A_y and A_z . From the definition of vector addition, it follows that (figure 3)

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

To spcify **A,** it is sufficient to specify its three components: (A_x, A_y , A_z). Therefore a vector can be defined in three dimensional space as an ordered set of three numbers.

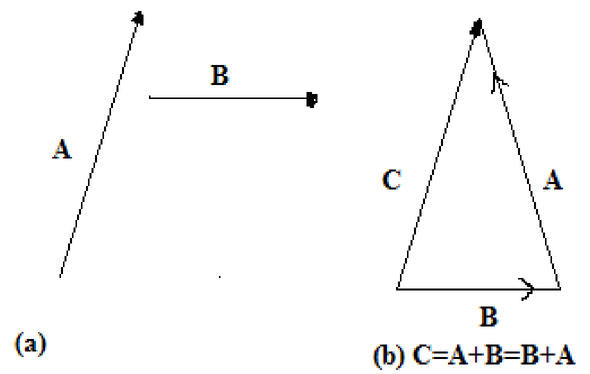


Figure 2: Addition of two vectors

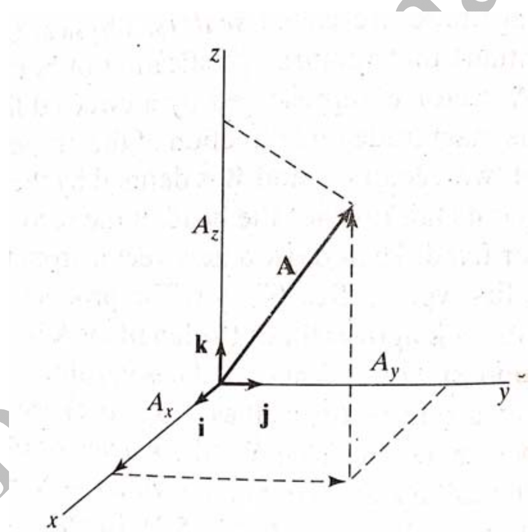


Figure 3: Unit vectors i,j,k and components of A

Two vectors **A** and **B** are equal if and only if all the corresponding components are equal:

$A_x = B_x$, $A_y = B_y$, $A_z = B_z$ .Hence a vector equation is equivalent to three scalar equations.

To add two vectors analytically, we add corresponding components:

A+B= $A_x\,i + A_y\,j + A_z\,k + B_x\,i + B_y\,j + B_z\,k$

A+B= $(A_x + B_x)\,i + (A_y + B_y)\,j + (A_z + B_z)\,k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$

Also, if c is a scalar, then $\quad cA = cA_x\,i + cA_y\,j + cA_z\,k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$

Angular Momentum

The **magnitude** A of a vector **A** is its length and is therefore a scalar. Often the notation $|A|$ is used for the magnitude of **A.**

The **dot product** or scalar product **A.B** of two vectors is defined by

$$\mathbf{A.B}=|\mathbf{A}|\,|\mathbf{B}|\cos\theta=\mathbf{B.A} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

where θ is the angle between the vectors. The dot product, being the product of three scalars, is a scalar. Note that $|\mathbf{A}|\cos\theta$ is the projection of **A** on **B.** From the definition of vector addition, it follows that the projection of the vector **A+B** on some vector **C** is the sum of the projections of **A** and of **B** on **C.** Hence

$$\mathbf{(A+B).C=A.C+B.C} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Since the three unit vectors I, j and k are each of unit length and are mutually perpendicular, we have

$$\mathbf{i.i=j.j=k.k}=\cos 0=1, \qquad \mathbf{ij=j.k=k.i}=\cos(\pi/2)=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Using eqn.(21) and the distributive law eqn.(20) , we have

$$\mathbf{A.B}=(A_x\,i+A_y\,j+A_z\,k).(B_x\,i+B_y\,j+B_z\,k)$$

$$\mathbf{A.B}=A_x B_x+A_y B_y+A_z B_z \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Where six of the nine terms in the dot product are zero.

Consider the dot product of a vector with itself. From eqn. (19) we have

$$\mathbf{A\,.\,A}=|\mathbf{A}|^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

Using eqn (22), we therefore have

$$|\mathbf{A}|=(A^2_x+A^2_y+A^2_z)^{1/2} \qquad \ldots\ldots\ldots\ldots\ldots (24)$$

For three –dimensional vectors, there is another type of product. The **cross product** or vector product **A X B** is a vector whose magnitude is

$$|\mathbf{A\,X\,B}|=|\mathbf{A}|\,|\mathbf{B}|.\sin\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

whose line segment is perpendicular to the plane defined by **A** and **B,** whose direction is such that **A , B** and **A X B** form a right-handed system (just as the x, y and z axes form a right handed system) . See Figure 4 . From the definition it follows that

$$\mathbf{BXA=-\,A\,X\,B} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Also, it can be shown that

$$\mathbf{A\,X\,(B+C)=A\,X\,B+A\,X\,C} \qquad \ldots\ldots\ldots\ldots (27)$$

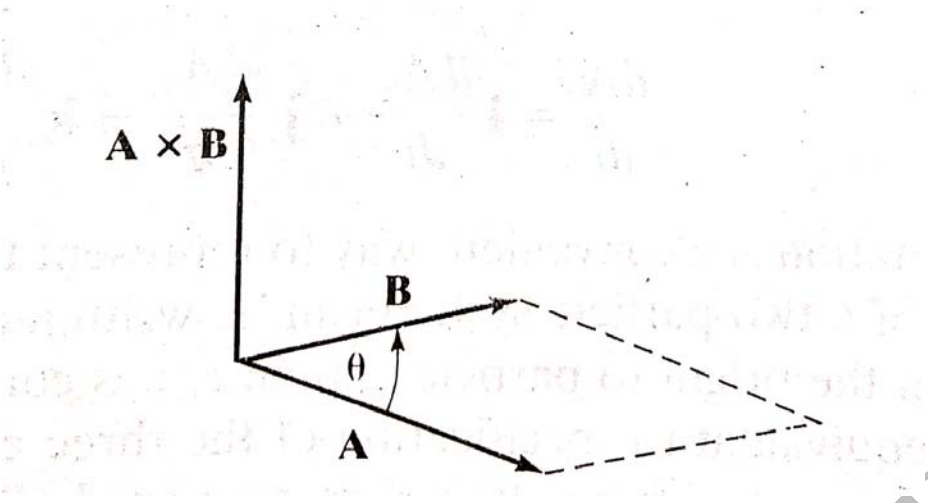


Figure 4: Cross product of two vectors

$i \times i = j \times j = k \times k = \sin 0 = 0$

$i \times j = k$, $j \times i = -k$, $j \times k = i$, $k \times j = -i$, $k \times i = j$, $i \times k = -j$

Using these equations and the distributive property eq. (27) , we find

A X B=$(A_x i + A_y j + A_z k) \times (B_x i + B_y j + B_z k)$

A X B=$(A_y B_z - A_z B_y) i + (A_z B_x - A_x B_z) j + (A_x B_y - A_y B_x) k$

As a memory aid , we can express the cross product as a determinant

$$\mathbf{A \times B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

…….. (28)

We defined the vector operator **del** as

$$\nabla \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

…….. (29)

The gradient of a function g(x,y,z) is defined as the result of operating on the function with del:

$$\textbf{grad}\ g(x,y,z) \equiv \nabla g(x,y,z) \equiv \mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

The gradient of a scalar function is a vector function. The vector $\nabla g(x,y,z)$ represents the spatial rate of change of the function g; the x component of ∇g is the rate of change of g with respect to x, and so on. It can be shown that the vector ∇g points in the direction in which the rate of change of g is greatest.

The relation between force and potential energy is

$$\mathbf{F} = -\nabla \mathbf{V}(x,y,z) = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z} \quad . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

Suppose that the components of a vector are each function of some parameter t; $A_x = A_x(t)$, $A_y = A_y(t)$, $A_z = A_z(t)$. We define the derivative of the vector with respect to t as

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt}$$

Vector notation is a conventional way to represent the variables of a function. The wave function of a two-particle system can be written as $\psi(x_1, y_1, z_1, x_2, y_2, z_2)$. If r_1 is the vector from the origin to particle 1, then r_1 has components x_1, y_1, z_1 and specification of r_1 is equivalent to specification of the three coordinates x_1, y_1, z_1. The same is true for the vector r_2 from the origin to particle 2 . Therefore we can write the wave function as $\psi(r_1, r_2)$. Vector notation is sometimes used in integrals. For example, the normalization integral over all space is often written as $\int\ldots\int |\psi(r_1,\ldots..r_n,t)|\, dr_1\ldots..d\, r_n$.

Vectors in n –Dimensional Space: For many purposes it is useful to generalize the definition of a vector to more than three dimension A vector **A** in three-dimensional space can be defined by its magnitude $|\mathbf{A}|$ and its direction, or can be defined by its three components (A_x, A_y, A_z) in a Cartesian Coordinate system. Therefore we can defined a three –dimensional vector as a set of three real number(A_x, A_y, A_z) in a particular order. A vector **B** in an n-dimensional real vector "space" (sometimes called a hyperspace) is defined as an ordered set of n real numbers ($B_1, B_2, \ldots\ldots\ldots B_n$) where $B_1, B_2, \ldots. B_n$ are the **components** of **B.**

The variables of a function are often denoted using n-dimensional vector notation. For example, instead of writing the wave function of a two particle system as $\psi(r_1, r_2)$, we can define **q** whose components are $q_1=x_1$, $q_2=y_1$, $q_3=z_1$, $q_4=x_2$, $q_5=y_2$, $q_6=z_2$ and write the wave function as $\psi(\mathbf{q})$. For an n-particle system, we can define **q** to have 3n components and write the wave function as $\psi(\mathbf{q})$ and the normalization integral over all space as $\int |\psi(\mathbf{q})|^2 d\mathbf{q}$.

Angular Momentum

Two n-dimensional vector are equal if all their corresponding components are equal; **B=C** if and only if $B_1=C_1$, $B_2=C_2$, ... $B_n=C_n$. Thus, in n-dimensional space, a vector equation is equivalent to n scalar equations. The sum of two n-dimensional vectors **B** and **D** is defined as the vector (B_1+D_1, B_2+D_2,…… B_n+D_n). The difference is defined similarly. The vector k**B** is defined as the vector (kB_1, kB_2,..... kB_n), where k is a scalar. In three dimensional space the vector k**A**, where k >0, all lie in the same direction. Just as the number (A_x, A_y, A_z) defined a point in the three- dimensional space, the numbers (B_1, B_2..... Bn) defined a point in n-dimensional space.

The **length (** or **magnitude)** $|\mathbf{B}|$ of an n-dimensional real vector is defined as

$$|\mathbf{B}| \equiv (\mathbf{B.B})^{1/2} \equiv (B^2_1 + B^2_2 + \ldots\ldots + B^2_n)^{1/2}$$

A vector whose length is 1 is said to be **normalized**.

The inner product (or scalar product) **B.G** of two real n-dimensional vector **B** and **G** is defined as the scalar

$$\mathbf{B.G} \equiv B_1G_1 + B_2G_2 + \ldots\ldots\ldots\ldots + B_nGn$$

If **B.G=0,** the vectors **B** and **G** are said to be **orthogonal**. The cosine of the angle θ between two n –dimensional vectors **B** and **G** is defined as $\cos\theta \equiv \mathbf{B.G}/|\mathbf{B}||\mathbf{G}|$. One can show that this definition makes cos θ lie in the range -1 to 1.

In three- dimensional space, the unit vectors **i**=(1,0,0), **j**=(0,1,0), **k**=(0,0,1) are mutually perpendicular. Also, any vector can be written as a linear combination of these three vectors (Eqn 16). In an n –dimensional real vector space, the unit vectors $\mathbf{e_1 \equiv (1,0,0,\ldots,0)}$, $\mathbf{e_1 \equiv (1,0,0,\ldots,0), e_2 \equiv (0,1,0,\ldots,0)}$,....., $\mathbf{e_n \equiv (0,0,0,\ldots,1)}$ are mutually orthogonal. Since the n –dimensional vector **B** equals $B_1e_1 + B_2e_2 + \ldots\ldots\ldots + B_ne_n$, any n-dimensional real vector can be written as a linear combination of the of the n unit vectors $\mathbf{e_1}$, $\mathbf{e_2}$..... $\mathbf{e_n}$. This set of n vectors is therefore said to be a **basis** for the n –dimensional real vector space. Since the vectors $\mathbf{e_1}$, $\mathbf{e_2}$..... $\mathbf{e_n}$ are orthogonal and normalized, they are an **orthonornal** basis for real vector space. The scalar product **B.** $\mathbf{e_1}$ gives the component of **B** in the direction of the basis vector $\mathbf{e_1}$. A vector space has many possible basis sets. Any set of n linearly independent real vectors can serve as a basis for the n –dimensional real vector space.

A three- dimensional vector can be specified by its three components or by its length and its direction. The direction can be specified by giving the three angles the vector makes with the positive halves of the x,y and z axes. These angles are the **direction angles** of the vector and lie in the range 0 to 180^0. However, the direction angle with the z axis is fixed once the other two direction angles have been given, so only two direction angles are independent. Thus a three –dimensional vector can be specified by its length and two direction angles. Similarly, in n-dimensional space, the direction angles between a vector and each unit vector $\mathbf{e_1}$,$\mathbf{e_2}$.....$\mathbf{e_n}$ can be

found from the above formula for the cosine of the angle between two vectors . An n – dimensional vector whose components are the first partial derivative of f:

$$\nabla f=(\partial f/\partial q_1)e_1+(\partial f/\partial q_2)e_2+.....+(\partial f/\partial q_n)e_n$$