Angular Momentum

We know from our previous discussion that the commutator $\hat{A}$ and $\hat{B}$, is defined as

$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

The following commutator identities are helpful in evaluating commutators:

$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ ....(1)

$[\hat{A}, \hat{A}^n] = 0$, n=1,2,3… ....(2)

$[k\hat{A}, \hat{B}] = [\hat{A}, k\hat{B}] = k[\hat{A}, \hat{B}]$ ....(3)

$[\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$, $[\hat{A}+\hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$ .... (4)

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}\hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$, $[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$ ....(5)

Where k is constant and the operator are assumed to be linear.

Example:

1. $[\partial/\partial x, x] = 1$

Proof: $[\partial/\partial x, x]f(x) = \partial/\partial x\{x.f(x)\} - x\partial/\partial x\{f(x)\}$

$= xf'(x) + f(x) - xf'(x) = 1.f(x)$

So, $[\partial/\partial x, x]f(x) = 1.f(x)$

Or, $[\partial/\partial x, x] = 1$ (proved) ....(6)

2.

$[\hat{x}, \hat{p}_x] = i\hbar$

Proof: $[\hat{x}, \hat{p}_x] = [x, \frac{\hbar}{i}\partial/\partial x] = \frac{\hbar}{i}[x, \partial/\partial x]$

$= -\frac{\hbar}{i}[\partial/\partial x, x] = -\frac{\hbar}{i}$

$[\hat{x}, \hat{p}_x] = i\hbar$

.... (7)

3.

$$[\hat{x},\hat{p}_x^2]=2\hbar^2\,\partial/\partial x$$

Proof: $[\hat{x},\hat{p}_x^2]=[\hat{x},\hat{p}_x]\,\hat{p}_x+\hat{p}_x\,[\hat{x},\hat{p}_x]$

$$=i\hbar\,\frac{\hbar}{i}\partial/\partial x+\frac{\hbar}{i}\partial/\partial x\;i\hbar$$

$$=2\hbar^2\,\partial/\partial x$$

……………………….(8)

4.

$$[\hat{x},\hat{H}]=[\hat{x},\hat{T}+\hat{V}]=[\hat{x},\hat{T}]+[\hat{x},\hat{V}(x,y,z)]=[\hat{x},\hat{T}]$$

$$=[x,(1/2m)(\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2)]$$

$$=(1/2m)[\hat{x},\hat{p}_x^2]+(1/2m)[\hat{x},\hat{p}_y^2]+(1/2m)[\hat{x},\hat{p}_z^2]$$

$$=(1/2m).\,2\hbar^2\,\partial/\partial x+0+0$$

$$[\hat{x},\hat{H}]=\frac{\hbar^2}{2m}\partial/\partial x=\frac{i\hbar}{m}\hat{p}_x\quad\text{(Proved)}$$

………………………… (9)

The above commutators have important physical consequences . Since $[\hat{x},\hat{p}_x]\neq 0$, we cannot expect the state function to be simultaneously an eigenfunction of $\hat{x}$ and of $\hat{p}_x$. Hence we cannot simultaneously assign definite values to x and p_x , in agreement with the uncertinity principle. Since $\hat{x}$ and $\hat{H}$ do not commute, we cannot expect to assign definite values to the energy and x coordinate at the same time . A stationary state (which has a definite energy) shows a spread of possible values for x, the probabilities for observing various values of x being given by Born postulate.

For a state function ψ that is not an eigen function of of Â, we get various possible outcomes when we measure A in identical system. We want some measure of the of the spread or dispersion in the set of observed values A_i. If <A> is the average of these values, then the deviation of each measurement from the average is A_i-<A>. If we averaged all the deviations , we would get zero, since positive and negative deviations would cancel. Hence to make all