DUMKAL COLLEGE

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Department of Mathematics

Class Note

B.Sc. Mathematics (Major)
Semester VI
Course Code: MATH-M-T-09
Course Title: Linear Programming Problems & Game Theory

Unit 1: Convex Sets, Basic Solutions, and Reduction to B.F.S.

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CLASS NOTE – Convex Sets and Basic Solutions

Introduction

In the study of linear programming problems, the concept of convex sets plays a fundamental role. The feasible region of a linear programming problem is always a convex set. Moreover, the optimal solution, when it exists, is found at an extreme point of this convex set. These extreme points correspond to basic feasible solutions. This note discusses convex sets, basic solutions, non-basic solutions, and the procedure for obtaining a basic feasible solution from a given basic solution.

Convex Sets

Definition 0.1. *A subset S of $\mathbb{R}^n$ is said to be a* ***convex set*** *if for any two points $x_1, x_2 \in S$ and for any real number λ satisfying $0 \leq \lambda \leq 1$, the point*

$$x = \lambda x_1 + (1 - \lambda)x_2$$

also belongs to S. In other words, the line segment joining any two points of S lies entirely within S.

Definition 0.2. *A point x is called a* ***convex combination*** *of the points $x_1, x_2, \ldots, x_k$ if there exist non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\sum_{i=1}^{k} \lambda_i = 1 \quad \text{and} \quad x = \sum_{i=1}^{k} \lambda_i x_i.$$

Definition 0.3. *A point x in a convex set S is called an* ***extreme point*** *(or vertex) of S if it cannot be expressed as a convex combination of two distinct points of S. Equivalently, if $x = \lambda x_1 + (1 - \lambda)x_2$ for some $x_1, x_2 \in S$ and $0 < \lambda < 1$, then $x_1 = x_2 = x$.*

Theorem 0.1. *The feasible region of a linear programming problem is a convex set.*

Proof. Consider the linear programming problem in standard form:

$$Ax = b, \quad x \geq 0.$$

Let $S = \{x : Ax = b, x \geq 0\}$. Take any two points $x_1, x_2 \in S$. Then $Ax_1 = b$, $Ax_2 = b$, and $x_1 \geq 0$, $x_2 \geq 0$. For any $\lambda \in [0, 1]$, consider $x = \lambda x_1 + (1 - \lambda)x_2$. Then

$$Ax = A(\lambda x_1 + (1 - \lambda)x_2) = \lambda Ax_1 + (1 - \lambda)Ax_2 = \lambda b + (1 - \lambda)b = b.$$

Also, since $x_1 \geq 0$ and $x_2 \geq 0$ and $\lambda, 1 - \lambda \geq 0$, we have $x \geq 0$. Therefore, $x \in S$. Hence, S is convex. □

Theorem 0.2. *The intersection of two convex sets is convex.*

Proof. Let S_1 and S_2 be convex subsets of $\mathbb{R}^n$. Let $x, y \in S_1 \cap S_2$. Then $x, y \in S_1$ and $x, y \in S_2$. Since S_1 is convex, for any $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y$ belongs to S_1. Similarly, because S_2 is convex, the same point belongs to S_2. Therefore, $\lambda x + (1 - \lambda)y \in S_1 \cap S_2$. Hence, $S_1 \cap S_2$ is convex. □

Theorem 0.3. *The intersection of any collection of convex sets is convex.*

Proof. Let $\{S_i\}_{i \in I}$ be a collection of convex sets. Let $S = \bigcap_{i \in I} S_i$. Take any $x, y \in S$. Then $x, y \in S_i$ for every $i \in I$. Since each S_i is convex, for any $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y$ belongs to S_i for every $i \in I$. Hence, $\lambda x + (1 - \lambda)y \in S$. Therefore, S is convex. □

Theorem 0.4. *The set of all convex combinations of a finite set of points is convex.*

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a finite set of points in $\mathbb{R}^n$. Let C be the set of all convex combinations of points in S. Take any two points $u, v \in C$. Then

$$u = \sum_{i=1}^{k} \alpha_i x_i, \quad v = \sum_{i=1}^{k} \beta_i x_i,$$

where $\alpha_i, \beta_i \geq 0$ and $\sum \alpha_i = \sum \beta_i = 1$. For any $\lambda \in [0, 1]$,

$$\lambda u + (1 - \lambda)v = \lambda \sum_{i=1}^{k} \alpha_i x_i + (1 - \lambda) \sum_{i=1}^{k} \beta_i x_i = \sum_{i=1}^{k} [\lambda \alpha_i + (1 - \lambda)\beta_i] x_i.$$

The coefficients $\lambda \alpha_i + (1 - \lambda)\beta_i$ are non-negative because $\lambda, \alpha_i, \beta_i \geq 0$. Moreover,

$$\sum_{i=1}^{k} [\lambda \alpha_i + (1 - \lambda)\beta_i] = \lambda \sum_{i=1}^{k} \alpha_i + (1 - \lambda) \sum_{i=1}^{k} \beta_i = \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.$$

Hence, $\lambda u + (1 - \lambda)v$ is also a convex combination of points in S, so it belongs to C. Therefore, C is convex. □

Theorem 0.5. *A convex combination of convex combinations is again a convex combination.*

Proof. Let $x_1, x_2, \ldots, x_k$ be points in a convex set. Suppose $y = \sum_{i=1}^{k} \alpha_i x_i$ is a convex combination, so $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Similarly, let $z = \sum_{j=1}^{l} \beta_j y_j$ be a convex combination of points y_j, each of which is itself a convex combination of the x_i. Then each $y_j = \sum_{i=1}^{k} \gamma_{ji} x_i$ with $\gamma_{ji} \geq 0$ and $\sum_i \gamma_{ji} = 1$. Substituting,

$$z = \sum_{j=1}^{l} \beta_j \left(\sum_{i=1}^{k} \gamma_{ji} x_i \right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{l} \beta_j \gamma_{ji} \right) x_i.$$

The coefficients $\sum_{j=1}^{l} \beta_j \gamma_{ji}$ are non-negative and

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \beta_j \gamma_{ji} = \sum_{j=1}^{l} \beta_j \left(\sum_{i=1}^{k} \gamma_{ji} \right) = \sum_{j=1}^{l} \beta_j = 1.$$

Thus, z is a convex combination of the original points x_i. □

Standard Form of a Linear Programming Problem

For the discussion of basic solutions, the linear programming problem is written in the following standard form:

$$\begin{aligned} \text{Minimize (or Maximize)} \quad & Z = c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix of rank m (with $m < n$), b is an $m \times 1$ vector with non-negative entries, c is an $n \times 1$ cost vector, and x is an $n \times 1$ vector of decision variables.

Basic Solutions

Definition 0.4. *Consider the system $Ax = b$ with A of size $m \times n$ and rank m. A **basic solution** is obtained by setting $n - m$ variables equal to zero and solving the remaining m equations for the remaining m variables, provided that the coefficient matrix of these m variables is non-singular.*

More formally:

1. Select m columns of A that are linearly independent. The corresponding m variables are called **basic variables**. The remaining $n - m$ variables are called **non-basic variables**.
2. Set all non-basic variables to zero.
3. Solve the system $A_B x_B = b$ for the basic variables, where A_B is the $m \times m$ matrix formed by the selected columns.

Definition 0.5. *A basic solution is called a **basic feasible solution** if all its basic variables satisfy the non-negativity condition, i.e., $x_B \geq 0$.*

Definition 0.6. *A basic solution is said to be **non-degenerate** if all the basic variables are strictly positive. If at least one basic variable is zero, the basic solution is called **degenerate**.*

Non-basic Solutions

Definition 0.7. *A **non-basic solution** is any solution of the system $Ax = b$ that is not a basic solution. In a non-basic solution, more than $n - m$ variables may be non-zero, or the chosen set of m variables does not yield a non-singular coefficient matrix.*

Reduction of a Basic Feasible Solution from a Basic Solution

Not every basic solution is feasible because some basic variables may be negative. The following procedure describes how to obtain a basic feasible solution from a given basic solution that is not feasible.

1. Start with a basic solution that may have some negative basic variables.
2. Identify a basic variable that is negative. This variable will be removed from the basis.
3. Choose a non-basic variable to enter the basis using a suitable ratio test so that the new basic solution moves towards feasibility.
4. Perform a pivot operation to exchange the leaving basic variable with the entering non-basic variable.
5. Repeat the process until all basic variables are non-negative, thereby obtaining a basic feasible solution.

Lemma 0.1. *If a basic solution has at least one negative basic variable, then there exists a pivot operation that either produces a new basic solution with a larger number of non-negative basic variables or identifies that the feasible region is empty.*

Proof. Let the current basic solution be given by $x_B = A_B^{-1}b$. Suppose that the p-th basic variable, denoted by x_{B_p}, is negative. Consider the p-th row of the canonical form:

$$x_{B_p} + \sum_{j \in N} \bar{a}_{pj} x_j = \bar{b}_p,$$

where $\bar{b}_p = x_{B_p} < 0$ and N is the set of indices of non-basic variables. If all $\bar{a}_{pj} \geq 0$ for all non-basic j, then the equation becomes

$$x_{B_p} = \bar{b}_p - \sum_{j \in N} \bar{a}_{pj} x_j \leq \bar{b}_p < 0,$$

since the sum is non-negative. This means that x_{B_p} remains negative for any non-negative choice of non-basic variables, implying that the feasible region is empty. If there exists a non-basic index q such that $\bar{a}_{pq} < 0$, then we can perform a pivot on $\bar{a}_{pq}$. The entering variable x_q increases from zero, and the leaving variable x_{B_p} decreases in magnitude. The pivot produces a new basic solution. The ratio test ensures that the new basic variables are non-negative. Thus, the number of negative basic variables either decreases or the algorithm terminates with the conclusion that no feasible solution exists. $\square$

Solved Problems

Question: 1. Define convex set. Show that the set $\{x : Ax \leq b, x \geq 0\}$ is convex.

Solution: A subset S of $\mathbb{R}^n$ is called a convex set if for any two points $x_1, x_2 \in S$ and for any $\lambda \in [0,1]$, the point $\lambda x_1 + (1-\lambda)x_2$ belongs to S.

Let $S = \{x : Ax \leq b, x \geq 0\}$. Take any two points $x_1, x_2 \in S$. Then $Ax_1 \leq b$, $Ax_2 \leq b$, and $x_1 \geq 0$, $x_2 \geq 0$. For any $\lambda \in [0,1]$, consider $x = \lambda x_1 + (1-\lambda)x_2$. Then

$$Ax = A(\lambda x_1 + (1-\lambda)x_2) = \lambda A x_1 + (1-\lambda)Ax_2 \leq \lambda b + (1-\lambda)b = b.$$

Also, since $x_1 \geq 0$ and $x_2 \geq 0$ and $\lambda, 1-\lambda \geq 0$, we have $x \geq 0$. Therefore, $x \in S$. Hence, S is convex.

Question: 2. Find all basic solutions of the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4, \\ 2x_1 + x_2 + x_4 &= 5, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Identify the basic feasible solutions.

Solution: Here $m = 2$ and $n = 4$. We consider all combinations of 2 columns from the coefficient matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

Basis $\{1, 2\}$**:** $A_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\det = 1 - 4 = -3 \neq 0$. Solving $x_1 + 2x_2 = 4$, $2x_1 + x_2 = 5$ gives $x_1 = 2$, $x_2 = 1$. Solution: $(2, 1, 0, 0)$ — feasible.

Basis $\{1, 3\}$**:** $A_B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, $\det = -2 \neq 0$. Solving gives $x_1 = 2.5$, $x_3 = 1.5$. Solution: $(2.5, 0, 1.5, 0)$ — feasible.

Basis $\{1, 4\}$**:** $A_B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_1 = 4$, $x_4 = -3$. Solution: $(4, 0, 0, -3)$ — not feasible.

Basis $\{2, 3\}$**:** $A_B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $\det = -1 \neq 0$. Solving gives $x_2 = -0.5$, $x_3 = 5$. Solution: $(0, -0.5, 5, 0)$ — not feasible.

Basis $\{2, 4\}$**:** $A_B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $\det = 2 \neq 0$. Solving gives $x_2 = 2$, $x_4 = 3$. Solution: $(0, 2, 0, 3)$ — feasible.

Basis $\{3, 4\}$**:** $A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_3 = 4$, $x_4 = 5$. Solution: $(0, 0, 4, 5)$ — feasible.

Thus, the basic solutions are $(2, 1, 0, 0)$, $(2.5, 0, 1.5, 0)$, $(4, 0, 0, -3)$, $(0, -0.5, 5, 0)$, $(0, 2, 0, 3)$, and $(0, 0, 4, 5)$. The basic feasible solutions are $(2, 1, 0, 0)$, $(2.5, 0, 1.5, 0)$, $(0, 2, 0, 3)$, and $(0, 0, 4, 5)$.

Question: 3. Find all basic solutions of the system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 3, \\ x_1 - x_2 + x_4 &= 1, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Identify the basic feasible solutions.

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$. We consider all combinations of 2 columns.

Basis $\{1, 2\}$**:** $A_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\det = -2 \neq 0$. Solving $x_1 + x_2 = 3$, $x_1 - x_2 = 1$ gives $x_1 = 2$, $x_2 = 1$. Solution: $(2, 1, 0, 0)$ — feasible.

Basis $\{1, 3\}$**:** $A_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\det = -1 \neq 0$. Solving gives $x_1 = 1$, $x_3 = 2$. Solution: $(1, 0, 2, 0)$ — feasible.

Basis $\{1,4\}$: $A_B = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_1 = 3$, $x_4 = -2$. Solution: $(3,0,0,-2)$ — not feasible.

Basis $\{2,3\}$: $A_B = \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_2 = -1$, $x_3 = 4$. Solution: $(0,-1,4,0)$ — not feasible.

Basis $\{2,4\}$: $A_B = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_2 = 3$, $x_4 = 4$. Solution: $(0,3,0,4)$ — feasible.

Basis $\{3,4\}$: $A_B = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_3 = 3$, $x_4 = 1$. Solution: $(0,0,3,1)$ — feasible.

Thus, the basic solutions are $(2,1,0,0)$, $(1,0,2,0)$, $(3,0,0,-2)$, $(0,-1,4,0)$, $(0,3,0,4)$, $(0,0,3,1)$. The basic feasible solutions are $(2,1,0,0)$, $(1,0,2,0)$, $(0,3,0,4)$, and $(0,0,3,1)$.

Question: 4. Find all basic solutions of the system:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3,\\ 2x_1 + x_2 + x_4 &= 4,\\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Identify the basic feasible solutions.

Solution: The coefficient matrix is $A = \begin{pmatrix}1 & 1 & 2 & 0\\ 2 & 1 & 0 & 1\end{pmatrix}$. We consider all combinations of 2 columns.

Basis $\{1,2\}$: $A_B = \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$, $\det = 1 - 2 = -1 \neq 0$. Solving $x_1 + x_2 = 3, 2x_1 + x_2 = 4$ gives $x_1 = 1$, $x_2 = 2$. Solution: $(1,2,0,0)$ — feasible.

Basis $\{1,3\}$: $A_B = \begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}$, $\det = -4 \neq 0$. Solving gives $x_1 = 2$, $x_3 = 0.5$. Solution: $(2,0,0.5,0)$ — feasible.

Basis $\{1,4\}$: $A_B = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_1 = 3$, $x_4 = -2$. Solution: $(3,0,0,-2)$ — not feasible.

Basis $\{2,3\}$: $A_B = \begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}$, $\det = -2 \neq 0$. Solving gives $x_2 = 4$, $x_3 = -1$. Solution: $(0,4,-1,0)$ — not feasible.

Basis $\{2,4\}$: $A_B = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_2 = 3$, $x_4 = 1$. Solution: $(0,3,0,1)$ — feasible.

Basis $\{3,4\}$: $A_B = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, $\det = 2 \neq 0$. Solving gives $x_3 = 1.5$, $x_4 = 4$. Solution: $(0,0,1.5,4)$ — feasible.

Thus, the basic feasible solutions are $(1,2,0,0)$, $(2,0,0.5,0)$, $(0,3,0,1)$, and $(0,0,1.5,4)$.

Question: 5. Find all basic feasible solutions of the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5,\\ x_1 + x_2 + x_4 &= 3,\\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Solution: The coefficient matrix is $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$. We consider all combinations of 2 columns.

Basis $\{1,2\}$**:** $A_B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\det = 1 - 2 = -1 \neq 0$. Solving $x_1 + 2x_2 = 5$, $x_1 + x_2 = 3$ gives $x_1 = 1$, $x_2 = 2$. Solution: $(1,2,0,0)$ — feasible.

Basis $\{1,3\}$**:** $A_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\det = -1 \neq 0$. Solving gives $x_1 = 3$, $x_3 = 2$. Solution: $(3,0,2,0)$ — feasible.

Basis $\{1,4\}$**:** $A_B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_1 = 5$, $x_4 = -2$. Solution: $(5,0,0,-2)$ — not feasible.

Basis $\{2,3\}$**:** $A_B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $\det = -1 \neq 0$. Solving gives $x_2 = 3$, $x_3 = -1$. Solution: $(0,3,-1,0)$ — not feasible.

Basis $\{2,4\}$**:** $A_B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, $\det = 2 \neq 0$. Solving gives $x_2 = 2.5$, $x_4 = 0.5$. Solution: $(0,2.5,0,0.5)$ — feasible.

Basis $\{3,4\}$**:** $A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\det = 1 \neq 0$. Solving gives $x_3 = 5$, $x_4 = 3$. Solution: $(0,0,5,3)$ — feasible.

Thus, the basic feasible solutions are $(1,2,0,0)$, $(3,0,2,0)$, $(0,2.5,0,0.5)$, and $(0,0,5,3)$.

Question: 6. Reduce the following basic solution to a basic feasible solution:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4, \\ 2x_1 + x_2 + x_4 &= 5, \end{aligned}$$

with basic variables x_1 and x_4.

Solution: The basic solution with basic variables x_1 and x_4 is obtained as follows. From the equations, solving for x_1 and x_4 in terms of x_2 and x_3: From the first equation, $x_1 = 4 - 2x_2 - x_3$. Substituting into the second equation:

$$2(4 - 2x_2 - x_3) + x_2 + x_4 = 5 \Rightarrow 8 - 4x_2 - 2x_3 + x_2 + x_4 = 5 \Rightarrow 8 - 3x_2 - 2x_3 + x_4 = 5 \Rightarrow x_4 = -3 +$$

Set $x_2 = 0$, $x_3 = 0$ to get the basic solution: $x_1 = 4$, $x_4 = -3$. Thus, the basic solution is $(4,0,0,-3)$. It is not feasible because $x_4 = -3 < 0$.

To reduce it to a basic feasible solution, we note that x_4 is negative. The coefficients of x_2 and x_3 in the expression for x_4 are both positive. Choose x_2 as the entering variable. From the expression for x_1, we have $x_1 = 4 - 2x_2$. For $x_1 \geq 0$, we need $x_2 \leq 2$. From the expression for x_4, we have $x_4 = -3 + 3x_2$. For $x_4 \geq 0$, we need $x_2 \geq 1$. The smallest x_2 that makes $x_4 = 0$ is $x_2 = 1$. At $x_2 = 1$, $x_1 = 4 - 2 = 2$, and $x_4 = 0$. The new basic variables are x_1 and x_2. The new basic feasible solution is $(2,1,0,0)$.

Question: 7. Reduce the following basic solution to a basic feasible solution:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2, \\ 2x_1 + x_2 + x_4 &= 1, \end{aligned}$$

with basic variables x_1 and x_4.

Solution: The basic solution with basic variables x_1 and x_4 is obtained as follows. From the equations, solving for x_1 and x_4 in terms of x_2 and x_3: From the first equation, $x_1 = 2 - x_2 - x_3$. Substituting into the second equation:

$2(2-x_2-x_3)+x_2+x_4 = 1 \Rightarrow 4-2x_2-2x_3+x_2+x_4 = 1 \Rightarrow 4-x_2-2x_3+x_4 = 1 \Rightarrow x_4 = -3+x_2$

Set $x_2 = 0$, $x_3 = 0$ to get the basic solution: $x_1 = 2$, $x_4 = -3$. Thus, the basic solution is $(2, 0, 0, -3)$. It is not feasible because $x_4 = -3 < 0$.

To reduce it to a basic feasible solution, we note that x_4 is negative. The coefficients of x_2 and x_3 in the expression for x_4 are both positive. Choose x_2 as the entering variable. From the expression for x_1, we have $x_1 = 2 - x_2$. For $x_1 \geq 0$, we need $x_2 \leq 2$. From the expression for x_4, we have $x_4 = -3 + x_2$. For $x_4 \geq 0$, we need $x_2 \geq 3$. No x_2 satisfies both. Try x_3 as entering variable: From $x_1 \geq 0$: $2 - x_3 \geq 0 \Rightarrow x_3 \leq 2$. From $x_4 \geq 0$: $-3 + 2x_3 \geq 0 \Rightarrow x_3 \geq 1.5$. Choose $x_3 = 1.5$. Then $x_1 = 2 - 1.5 = 0.5$, $x_4 = -3 + 2(1.5) = 0$. The new basic variables are x_1 and x_3. The new basic feasible solution is $(0.5, 0, 1.5, 0)$.

Question: 8. Prove that every basic feasible solution is an extreme point of the feasible region.

Solution: Let x be a basic feasible solution of the system $Ax = b$, $x \geq 0$. Let the basic variables be $x_{B_1}, x_{B_2}, \ldots, x_{B_m}$ and the corresponding basic matrix be A_B. Suppose, for contradiction, that x is not an extreme point. Then there exist distinct points y, z in the feasible region and a scalar λ with $0 < \lambda < 1$ such that $x = \lambda y + (1-\lambda)z$. For any non-basic variable x_j, we have $x_j = 0$. Since $y_j \geq 0$ and $z_j \geq 0$, the equation $0 = \lambda y_j + (1-\lambda)z_j$ implies $y_j = z_j = 0$. Thus, all non-basic variables are zero in both y and z. Therefore, y and z are determined by their basic components. Since $Ay = b$ and $Az = b$, we have $A_B y_B = b$ and $A_B z_B = b$. Because A_B is invertible, $y_B = z_B = A_B^{-1} b = x_B$. Hence, $y = z = x$, contradicting the assumption that y and z are distinct. Therefore, x is an extreme point.

Question: 9. Prove that if a basic solution is degenerate, then the corresponding basic matrix is not uniquely determined.

Solution: A basic solution is called degenerate if at least one basic variable is zero. Suppose that a basic solution x is degenerate. Then there exists a basic variable, say $x_{B_p} = 0$. Consider the set of columns corresponding to the positive components of x. Since $x_{B_p} = 0$, this set does not include the p-th column of the basis. However, the p-th column can be replaced by any other column that is linearly independent of the remaining basic columns while still maintaining a basis. This is because the zero basic variable can be exchanged with a non-basic variable without changing the solution. Thus, there are multiple choices of the basis that yield the same basic solution. Hence, the basic matrix is not uniquely determined.

Question: 10. Prove that the set of all convex combinations of a finite set of points is convex.

Solution: Let $S = \{x_1, x_2, \ldots, x_k\}$ be a finite set of points in $\mathbb{R}^n$. Let C be the set of all convex combinations of points in S. Take any two points $u, v \in C$. Then

$$u = \sum_{i=1}^{k} \alpha_i x_i, \quad v = \sum_{i=1}^{k} \beta_i x_i,$$

where $\alpha_i, \beta_i \geq 0$ and $\sum \alpha_i = \sum \beta_i = 1$. For any $\lambda \in [0,1]$,

$$\lambda u + (1-\lambda)v = \lambda \sum_{i=1}^{k} \alpha_i x_i + (1-\lambda) \sum_{i=1}^{k} \beta_i x_i = \sum_{i=1}^{k} [\lambda \alpha_i + (1-\lambda)\beta_i] x_i.$$

The coefficients $\lambda \alpha_i + (1-\lambda)\beta_i$ are non-negative because $\lambda, \alpha_i, \beta_i \geq 0$. Moreover,

$$\sum_{i=1}^{k} [\lambda \alpha_i + (1-\lambda)\beta_i] = \lambda \sum_{i=1}^{k} \alpha_i + (1-\lambda) \sum_{i=1}^{k} \beta_i = \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.$$

Hence, $\lambda u + (1-\lambda)v$ is also a convex combination of points in S, so it belongs to C. Therefore, C is convex.

Summary Table

Term	Definition
Convex set	Contains the line segment between any two points
Extreme point	Cannot be expressed as a convex combination of other points
Basic solution	Solution obtained by setting $n - m$ variables to zero
Basic feasible solution (B.F.S.)	Basic solution with all variables non-negative
Non-basic solution	Any solution that is not a basic solution
Degenerate B.F.S.	Basic feasible solution with at least one zero basic variable

Additional Problems

Q1. Define convex combination. Show that a convex combination of convex combinations is again a convex combination.

Q2. Find all basic feasible solutions of the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6, \\ 2x_1 + x_2 + x_4 &= 5, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Q3. Find all basic solutions of the system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4, \\ x_1 + 2x_2 + x_4 &= 5, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Q4. Prove that a basic feasible solution cannot have more than m positive components.

Q5. Reduce the following basic solution to a basic feasible solution:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3, \\ x_1 + 2x_2 + x_4 &= 4, \end{aligned}$$

with basic variables x_1 and x_4.

Q6. Show that the feasible region of a linear programming problem is a convex polyhedron.

Q7. Give an example of a degenerate basic feasible solution.

Q8. Prove that if the feasible region is bounded, then every basic feasible solution is an extreme point.
