DUMKAL COLLEGE
Dumkal, Murshidabad – 742406
West Bengal, India

Department of Mathematics

Class Note

B.Sc. Mathematics (Major)
Semester II
Course Code: MATH-T-02
Course Title: Algebra-I

Unit 1: Complex Numbers

Prepared by:
Dr. Tanchar Molla
Assistant Professor
Department of Mathematics
Dumkal College, Dumkal

April 6, 2026

CLASS NOTE – Complex Numbers

Introduction

A complex number is an expression of the form $z = x + iy$, where x and y are real numbers and i satisfies $i^2 = -1$. The real number x is called the real part of z and is denoted by $\mathrm{Re}(z)$. The real number y is called the imaginary part of z and is denoted by $\mathrm{Im}(z)$. The set of all complex numbers is denoted by $\mathbb{C}$.

Polar Representation of Complex Numbers

Definition 0.1 (Modulus and Argument). *For a complex number $z = x + iy$, the modulus of z is defined as $|z| = \sqrt{x^2 + y^2}$. The argument of z, denoted by $\arg z$, is the angle θ such that $x = |z| \cos\theta$ and $y = |z| \sin\theta$. Thus, any non-zero complex number can be written in polar form as*

$$z = |z|(\cos\theta + i\sin\theta).$$

Definition 0.2 (Principal Argument). *The principal value of the argument, denoted by Arg z, is the unique value of θ satisfying $-\pi < \theta \leq \pi$.*

The Exponential Function $\exp(z)$

Definition 0.3 (Exponential Function). *For a complex number z, the exponential function $\exp(z)$ is defined by the power series*

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

This series converges absolutely for every complex number z.

Theorem 0.1 (Convergence of the Exponential Series). *The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for every complex number z.*

Proof. For any fixed z, applying the ratio test gives

$$\lim_{n\to\infty} \left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0 < 1.$$

Therefore, the series converges absolutely for all $z \in \mathbb{C}$. □

Theorem 0.2 (Properties of the Exponential Function). *The exponential function satisfies the following properties:*

(i) $\exp(0) = 1$.

(ii) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ *for all* $z_1, z_2 \in \mathbb{C}$.

(iii) $\exp(-z) = \frac{1}{\exp(z)}$.

(iv) $\exp(z)$ *is periodic with period* $2\pi i$*, that is,* $\exp(z + 2\pi i) = \exp(z)$.

(v) $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$.

(vi) $|\exp(z)| = e^{Re(z)}$.

(vii) $\overline{\exp(z)} = \exp(\overline{z})$.

Proof. The first property follows directly from the series definition. The second property follows from the series definition and the binomial theorem. The third property follows from the second property with $z_1 = z$ and $z_2 = -z$. The fourth property is proved using Euler's formula. The fifth property follows from the sixth because $|\exp(z)| = e^{\mathrm{Re}(z)} > 0$. The sixth property is proved using Euler's formula. □

Euler's Formula and the Notation e^z

Theorem 0.3 (Euler's Formula). *For any real number* θ*,*

$$\exp(i\theta) = \cos\theta + i\sin\theta.$$

Proof. Using the series definition,

$$\exp(i\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!}.$$

Separating the even and odd terms yields

$$\exp(i\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta.$$

□

Definition 0.4 (The Notation e^z). *For a complex number* $z = x + iy$*, we write* e^z *to denote* $\exp(z)$*. Thus,*

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Corollary 0.1. *Every non-zero complex number can be written in the exponential polar form*

$$z = re^{i\theta},$$

where $r = |z|$ *and* $\theta = \arg z$.

The Logarithm Function $\log z$

Now that the exponential function has been defined, its inverse, the logarithm, can be introduced.

Definition 0.5 (Logarithm of a Complex Number)**.** *For a non-zero complex number z, the logarithm of z is defined as the inverse of the exponential function. That is, $w = \log z$ if and only if $e^w = z$. Since the exponential function is periodic with period $2\pi i$, the logarithm is multi-valued. Explicitly,*

$$\log z = \ln |z| + i \arg z,$$

where $\arg z = \theta + 2k\pi$ for some integer k.

Proof. Let $z = re^{i\theta}$ with $r = |z| > 0$. Suppose $w = u + iv$ satisfies $e^w = z$. Then $e^u e^{iv} = re^{i\theta}$. Hence, $e^u = r$ so $u = \ln r$, and $e^{iv} = e^{i\theta}$ so $v = \theta + 2k\pi$ for some integer k. Therefore,

$$w = \ln r + i(\theta + 2k\pi) = \ln |z| + i \arg z.$$

□

Definition 0.6 (Principal Value of Logarithm)**.** *The principal value of the logarithm, denoted by Log z, is given by*

$$\mathrm{Log}\, z = \ln |z| + i\mathrm{Arg}\, z,$$

where Arg z is the principal argument satisfying $-\pi < \mathrm{Arg}\, z \leq \pi$.

Theorem 0.4 (Inverse Relationship Between Exponential and Logarithm)**.** *For any non-zero complex number z,*

$$\exp(\log z) = z,$$

and for any complex number z,

$$\log(\exp z) = z + 2k\pi i$$

for some integer k.

Theorem 0.5 (Properties of Logarithm)**.** *For non-zero complex numbers z_1 and z_2,*

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$ *as sets of values.*

(ii) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$ *as sets of values.*

(iii) $\log(z^n) = n \log z$ *for any integer n.*

Proof. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\log(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2k\pi) = (\ln r_1 + i\theta_1) + (\ln r_2 + i\theta_2) = \log z_1 + \log z_2.$$

The other properties follow similarly. □

De Moivre's Theorem

Theorem 0.6 (De Moivre's Theorem for Integer Indices). *For any integer n and any real number θ,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

In exponential form, this becomes $(e^{i\theta})^n = e^{in\theta}$.

Proof. Using the exponential form, $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$. This proof holds for all integers n. □

Theorem 0.7 (De Moivre's Theorem for Rational Indices). *If $n = p/q$ is a rational number in lowest terms, then*

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right),$$

for $k = 0, 1, 2, \ldots, q-1$. Thus, the expression yields q distinct values.

nth Roots of Unity

Definition 0.7 (nth Roots of Unity). *The nth roots of unity are the solutions of the equation $z^n = 1$, where n is a positive integer.*

Theorem 0.8. *The nth roots of unity are given by*

$$\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n},$$

for $k = 0, 1, 2, \ldots, n-1$. These n roots are distinct and lie on the unit circle $|z| = 1$.

Proof. Let $z = re^{i\theta}$. Then $z^n = r^n e^{in\theta} = 1 = e^{2k\pi i}$. Therefore, $r^n = 1$ so $r = 1$, and $n\theta = 2k\pi$ so $\theta = 2k\pi/n$. □

Corollary 0.2. *The sum of all nth roots of unity is zero, that is,*

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0,$$

where $\omega = e^{2\pi i/n}$.

Proof. The roots are precisely the roots of the polynomial equation $z^n - 1 = 0$. The sum of the roots equals the negative of the coefficient of z^{n-1}, which is zero. □

Example 0.1. *The cube roots of unity are the solutions of $z^3 = 1$. They are*

$$1, \quad \omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \omega^2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

These satisfy $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

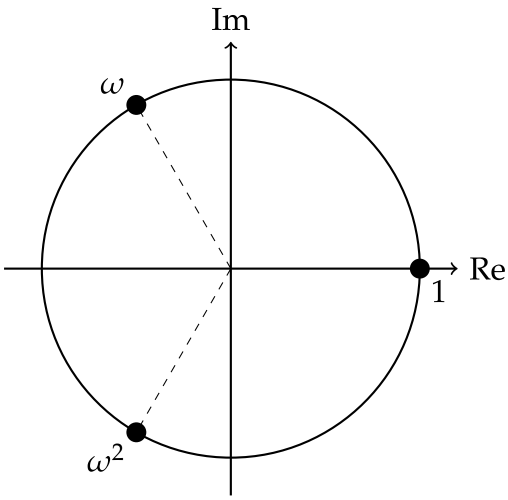

Figure 1: Cube Roots of Unity

Circular Functions of a Complex Variable

Definition 0.8. *For a complex number z, the sine and cosine functions are defined by*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The tangent, cotangent, secant, and cosecant functions are defined in the usual manner as

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Hyperbolic Functions of a Complex Variable

Definition 0.9. *For a complex number z, the hyperbolic sine and cosine functions are defined by*

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

The hyperbolic tangent, cotangent, secant, and cosecant functions are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad z = \frac{1}{\cosh z}, \quad z = \frac{1}{\sinh z}.$$

Theorem 0.9 (Relations Between Circular and Hyperbolic Functions)**.** *For any complex number z, the following relations hold:*

$$\cos(iz) = \cosh z, \quad \sin(iz) = i \sinh z,$$
$$\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z.$$

Proof. Using the definitions,

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z,$$

and

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i\left(\frac{e^{z} - e^{-z}}{2}\right) = i \sinh z.$$

The other relations are proved similarly. □

Definition of z^c for Complex z and c

Definition 0.10. *For a non-zero complex number z and any complex number c, we define*

$$z^c = e^{c \log z},$$

where $\log z$ *is the multi-valued logarithm. Consequently,* z^c *is generally multi-valued.*

Example 0.2. *Find all values of* i^i.
Solution: Write $i = e^{i\pi/2}$*. Then*

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z}.$$

Therefore,

$$i^i = e^{i \log i} = e^{i \cdot i(\pi/2 + 2k\pi)} = e^{-(\pi/2 + 2k\pi)}.$$

All these values are real numbers. The principal value, obtained by taking $k = 0$*, is* $e^{-\pi/2}$.

Solved Problems

Question: 1. Find all the fourth roots of unity and represent them on the complex plane.
Solution: The fourth roots of unity are the solutions of $z^4 = 1$. They are given by

$$\omega_k = e^{2k\pi i/4} = e^{k\pi i/2}, \quad k = 0, 1, 2, 3.$$

For $k = 0$, $\omega_0 = e^0 = 1$. For $k = 1$, $\omega_1 = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$. For $k = 2$, $\omega_2 = e^{i\pi} = \cos\pi + i\sin\pi = -1$. For $k = 3$, $\omega_3 = e^{3i\pi/2} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$. Thus, the four roots are $1, i, -1, -i$.

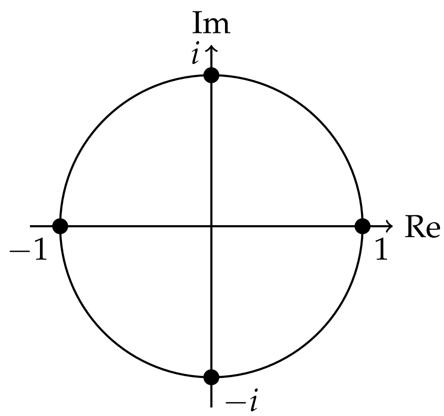

Figure 2: Fourth Roots of Unity

Question: 2. Using De Moivre's theorem, express $\cos 5\theta$ in terms of powers of $\cos\theta$.
Solution: By De Moivre's theorem,

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5.$$

Expanding the right-hand side using the binomial theorem gives

$$(\cos\theta + i\sin\theta)^5 = \sum_{r=0}^{5} \binom{5}{r} \cos^{5-r}\theta (i\sin\theta)^r.$$

The real part of this expansion gives $\cos 5\theta$. Taking the terms where r is even,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$ and $\sin^4\theta = (1-\cos^2\theta)^2$ yields

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta).$$

Simplifying,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta.$$

Therefore,

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Question: 3. Find the principal value of $\mathrm{Log}(1+i)$.
Solution: Let $z = 1+i$. Then $|z| = \sqrt{1^2+1^2} = \sqrt{2}$. The principal argument is $\mathrm{Arg}(z) = \tan^{-1}(1/1) = \frac{\pi}{4}$. Hence,

$$\mathrm{Log}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

Question: 4. Find all values of $(1+i)^{1/3}$.
Solution: First, write $1+i$ in polar form:

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i(\pi/4+2k\pi)}, \quad k \in \mathbb{Z}.$$

Then

$$(1+i)^{1/3} = (\sqrt{2})^{1/3}e^{i(\pi/12+2k\pi/3)} = 2^{1/6}e^{i(\pi/12+2k\pi/3)}, \quad k = 0,1,2.$$

The three distinct values are:

$$k=0: \quad 2^{1/6}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),$$

$$k=1: \quad 2^{1/6}\left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right) = 2^{1/6}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

$$k=2: \quad 2^{1/6}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right).$$

Question: 5. Prove that $\cosh^2 z - \sinh^2 z = 1$ for all complex z.
Solution: Using the definitions,

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Then

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2.$$

Expanding both squares gives

$$\cosh^2 z - \sinh^2 z = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = \frac{4}{4} = 1.$$

Question: 6. Find the real and imaginary parts of $\sin(1+i)$.

Solution: Using the formula $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ with $x = 1$ and $y = 1$,

$$\sin(1+i) = \sin 1 \cosh 1 + i \cos 1 \sinh 1.$$

Therefore, the real part is $\sin 1 \cosh 1$ and the imaginary part is $\cos 1 \sinh 1$.

Question: 7. Show that the sum of the nth roots of unity is zero.

Solution: Let $\omega = e^{2\pi i/n}$. The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Their sum is

$$S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}.$$

This is a geometric series with common ratio ω. Since $\omega \neq 1$,

$$S = \frac{1-\omega^n}{1-\omega} = \frac{1-1}{1-\omega} = 0.$$

Question: 8. Solve the equation $z^4 + z^2 + 1 = 0$.

Solution: Let $w = z^2$. Then the equation becomes $w^2 + w + 1 = 0$. Solving this quadratic equation yields

$$w = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = e^{\pm 2\pi i/3}.$$

Thus, $z^2 = e^{2\pi i/3}$ or $z^2 = e^{-2\pi i/3}$. Taking square roots,

$$z = \pm e^{\pi i/3} = \pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right),$$

$$z = \pm e^{-\pi i/3} = \pm\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).$$

The four solutions are $\pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

Question: 9. Find all values of i^i.

Solution: Write i in exponential form as $i = e^{i\pi/2}$. Then

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z}.$$

Therefore,

$$i^i = e^{i \log i} = e^{i \cdot i(\pi/2 + 2k\pi)} = e^{-(\pi/2+2k\pi)}.$$

All these values are real numbers. The principal value, corresponding to $k = 0$, is $e^{-\pi/2}$.

Question: 10. If $z = x + iy$, prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$.
Solution: Using $\sin z = \sin x \cosh y + i \cos x \sinh y$, we have

$$|\sin z|^2 = (\sin x \cosh y)^2 + (\cos x \sinh y)^2.$$

Since $\cosh^2 y = 1 + \sinh^2 y$ and $\cos^2 x = 1 - \sin^2 x$,

$$|\sin z|^2 = \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y.$$

Simplifying,

$$|\sin z|^2 = \sin^2 x + \sin^2 x \sinh^2 y + \sinh^2 y - \sin^2 x \sinh^2 y = \sin^2 x + \sinh^2 y.$$

Summary of Key Formulas

Exponential series	$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
Euler's formula	$e^{i\theta} = \cos\theta + i\sin\theta$
Polar form	$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$
De Moivre's theorem	$(e^{i\theta})^n = e^{in\theta}$
nth roots of unity	$e^{2k\pi i/n}, \quad k = 0, 1, \ldots, n-1$
Sum of roots of unity	$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$
$\cos z$	$\frac{e^{iz}+e^{-iz}}{2}$
$\sin z$	$\frac{e^{iz}-e^{-iz}}{2i}$
$\cosh z$	$\frac{e^{z}+e^{-z}}{2}$
$\sinh z$	$\frac{e^{z}-e^{-z}}{2}$
$\log z$	$\ln\|z\| + i \arg z$
z^c	$e^{c \log z}$

Practice Problems

Q1. Find all the sixth roots of unity and represent them on the complex plane.

Q2. Using De Moivre's theorem, express $\sin 4\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

Q3. Find the principal value of $\text{Log}(-1 - i)$.

Q4. Find all values of $(1 - i)^{1/4}$.

Q5. Prove that $\sin^2 z + \cos^2 z = 1$ for all complex z.

Q6. Find the real and imaginary parts of $\cos(2 + 3i)$.

Q7. Solve the equation $z^3 + 1 = 0$.

Q8. Find all values of $(1)^{1/5}$.

Q9. Prove that $|\cos z|^2 = \cos^2 x + \sinh^2 y$ where $z = x + iy$.

Q10. Show that $\tan^{-1}(i) = \frac{i}{2}\ln\left(\frac{1+i}{1-i}\right)$.

Common Mistakes

- The argument $\arg z$ is multi-valued, whereas the principal argument $\operatorname{Arg} z$ is unique and lies in the interval $(-\pi, \pi]$.
- De Moivre's theorem holds for integer exponents directly, but for rational exponents it yields multiple values.
- The exponential function satisfies $e^{z+2\pi i} = e^z$, which is the reason for the multi-valued nature of the logarithm.
- The logarithm of a complex number is multi-valued because the exponential function is periodic.
- The relations $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$ are often confused.
