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Department of Mathematics

Class Note

B.Sc. Mathematics (Major)
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Unit 1: Group Homomorphisms

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CLASS NOTE – Group Homomorphisms

Introduction

A homomorphism is a mapping between two groups that preserves the group structure. The study of homomorphisms is fundamental in group theory because it allows us to compare different groups and to understand the internal structure of a group by studying its images and kernels.

Key Concept: A group homomorphism is a function $\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. It preserves identities, inverses, and powers.

Definition of Group Homomorphism

Definition 0.1. *Let $(G, \cdot)$ and $(H, *)$ be two groups. A mapping $\phi : G \to H$ is called a **group homomorphism** (or simply a homomorphism) if*

$$\phi(a \cdot b) = \phi(a) * \phi(b) \quad \text{for all } a, b \in G.$$

In other words, ϕ preserves the group operation.

Definition 0.2.

- *A homomorphism $\phi : G \to H$ is called a **monomorphism** if it is injective (one-to-one).*
- *A homomorphism $\phi : G \to H$ is called an **epimorphism** if it is surjective (onto).*
- *A homomorphism $\phi : G \to H$ is called an **isomorphism** if it is bijective (both one-to-one and onto).*
- *A homomorphism $\phi : G \to G$ is called an **endomorphism** of G.*
- *An isomorphism $\phi : G \to G$ is called an **automorphism** of G.*

Examples of Group Homomorphisms

Example 0.1 (Trivial Homomorphism)**.** *Let G and H be any two groups. Define $\phi : G \to H$ by $\phi(g) = e_H$ for all $g \in G$. Then for any $a, b \in G$,*

$$\phi(ab) = e_H = e_H e_H = \phi(a)\phi(b).$$

*Thus, ϕ is a homomorphism, called the **trivial homomorphism**.*

Example 0.2 (Identity Homomorphism). *Let G be any group. Define $\phi : G \to G$ by $\phi(g) = g$ for all $g \in G$. Then $\phi(ab) = ab = \phi(a)\phi(b)$. This is called the* ***identity homomorphism*** *or* ***identity automorphism****.*

Example 0.3 (Inclusion Map). *Let H be a subgroup of G. Define $\phi : H \to G$ by $\phi(h) = h$ for all $h \in H$. This is a monomorphism called the* ***inclusion map****.*

Example 0.4 (Exponential Map). *Define $\phi : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ by $\phi(x) = e^x$. Then*

$$\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\phi(y).$$

Thus, ϕ is an isomorphism between the additive group of real numbers and the multiplicative group of positive real numbers.

Example 0.5 (Logarithm Map). *Define $\psi : (\mathbb{R}^+, \cdot) \to (\mathbb{R}, +)$ by $\psi(x) = \ln x$. Then*

$$\psi(xy) = \ln(xy) = \ln x + \ln y = \psi(x) + \psi(y).$$

This is the inverse of the exponential map and is also an isomorphism.

Example 0.6 (Determinant Map). *Let $G = GL(n, \mathbb{R})$ (the group of invertible $n \times n$ real matrices) and $H = (\mathbb{R}^*, \cdot)$ (the multiplicative group of non-zero real numbers). Define $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$ by $\phi(A) = \det(A)$. Then*

$$\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\phi(B).$$

Thus, ϕ is an epimorphism (surjective homomorphism). Its kernel is $SL(n, \mathbb{R})$, the group of matrices with determinant 1.

Example 0.7 (Sign Map). *Let $G = S_n$ (the symmetric group) and $H = \{1, -1\}$ under multiplication. Define $\phi : S_n \to \{1, -1\}$ by $\phi(\sigma) = sgn(\sigma)$ (the sign of the permutation). Then*

$$\phi(\sigma\tau) = sgn(\sigma\tau) = sgn(\sigma)sgn(\tau) = \phi(\sigma)\phi(\tau).$$

Thus, ϕ is an epimorphism. Its kernel is A_n, the alternating group.

Basic Properties of Homomorphisms

Theorem 0.1 (Properties of Homomorphisms). *Let $\phi : G \to H$ be a group homomorphism. Then*

- *(i)* $\phi(e_G) = e_H$ *(the identity element is preserved).*
- *(ii)* $\phi(a^{-1}) = [\phi(a)]^{-1}$ *for all $a \in G$ (inverses are preserved).*
- *(iii)* $\phi(a^n) = [\phi(a)]^n$ *for all $a \in G$ and all integers n (powers are preserved).*
- *(iv) If $a \in G$ has finite order n, then the order of $\phi(a)$ divides n.*
- *(v) The image $\phi(G)$ is a subgroup of H.*
- *(vi) The kernel* $\ker \phi = \{g \in G : \phi(g) = e_H\}$ *is a subgroup of G.*

Proof. (i) We have $\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G)$. Multiplying both sides on the left by $[\phi(e_G)]^{-1}$ (which exists in H), we get $e_H = \phi(e_G)$.

(ii) For any $a \in G$,

$$\phi(a)\phi(a^{-1}) = \phi(aa^{-1}) = \phi(e_G) = e_H.$$

Similarly, $\phi(a^{-1})\phi(a) = e_H$. Hence, $\phi(a^{-1})$ is the inverse of $\phi(a)$ in H. Therefore, $\phi(a^{-1}) = [\phi(a)]^{-1}$.

(iii) For $n > 0$, the statement follows by induction. For $n = 0$, $\phi(a^0) = \phi(e_G) = e_H = [\phi(a)]^0$. For $n < 0$, write $n = -m$ where $m > 0$. Then

$$\phi(a^n) = \phi((a^m)^{-1}) = [\phi(a^m)]^{-1} = [\phi(a)^m]^{-1} = \phi(a)^{-m} = \phi(a)^n.$$

(iv) Let $|a| = n$. Then $a^n = e_G$, so $\phi(a)^n = \phi(a^n) = \phi(e_G) = e_H$. Therefore, the order of $\phi(a)$ divides n.

(v) Since $e_H = \phi(e_G) \in \phi(G)$, $\phi(G)$ is non-empty. Let $\phi(a), \phi(b) \in \phi(G)$. Then $\phi(a)[\phi(b)]^{-1} = \phi(a)\phi(b^{-1}) = \phi(ab^{-1}) \in \phi(G)$. Hence, $\phi(G)$ is a subgroup of H.

(vi) Since $\phi(e_G) = e_H$, $e_G \in \ker\phi$. Let $a, b \in \ker\phi$. Then $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = e_H e_H = e_H$. Thus, $ab^{-1} \in \ker\phi$. Hence, $\ker\phi$ is a subgroup of G. ☐

Theorem: Kernel is a Normal Subgroup

Theorem 0.2. *Let $\phi : G \to H$ be a group homomorphism. Then $\ker\phi$ is a normal subgroup of G.*

Proof. From Theorem 0.1, we know that $\ker\phi$ is a subgroup of G. To prove normality, let $g \in G$ and $k \in \ker\phi$. Then

$$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g)e_H\phi(g)^{-1} = e_H.$$

Thus, $gkg^{-1} \in \ker\phi$. Hence, $g(\ker\phi)g^{-1} \subseteq \ker\phi$ for all $g \in G$. Therefore, $\ker\phi \triangleleft G$. ☐

Theorem: A Homomorphism is Injective if and only if Kernel is Trivial

Theorem 0.3. *A group homomorphism $\phi : G \to H$ is injective (one-to-one) if and only if $\ker\phi = \{e_G\}$.*

Proof. ($\Rightarrow$) Suppose ϕ is injective. Let $k \in \ker\phi$. Then $\phi(k) = e_H = \phi(e_G)$. Since ϕ is injective, $k = e_G$. Thus, $\ker\phi = \{e_G\}$.

($\Leftarrow$) Suppose $\ker\phi = \{e_G\}$. Let $a, b \in G$ such that $\phi(a) = \phi(b)$. Then $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)[\phi(b)]^{-1} = e_H$. Hence, $ab^{-1} \in \ker\phi = \{e_G\}$, so $ab^{-1} = e_G$ and $a = b$. Therefore, ϕ is injective. ☐

Theorem: Image of a Subgroup is a Subgroup

Theorem 0.4. *Let* $\phi : G \to H$ *be a group homomorphism.*

(a) If A *is a subgroup of* G*, then* $\phi(A)$ *is a subgroup of* H*.*

(b) If B *is a subgroup of* H*, then* $\phi^{-1}(B) = \{g \in G : \phi(g) \in B\}$ *is a subgroup of* G*.*

Proof. (a) Since $e_G \in A$, $\phi(e_G) = e_H \in \phi(A)$. Let $\phi(a_1), \phi(a_2) \in \phi(A)$. Then

$$\phi(a_1)[\phi(a_2)]^{-1} = \phi(a_1)\phi(a_2^{-1}) = \phi(a_1 a_2^{-1}) \in \phi(A),$$

because $a_1 a_2^{-1} \in A$. Hence, $\phi(A)$ is a subgroup of H.

(b) Since $\phi(e_G) = e_H \in B$, $e_G \in \phi^{-1}(B)$. Let $g_1, g_2 \in \phi^{-1}(B)$. Then $\phi(g_1), \phi(g_2) \in B$. Now,

$$\phi(g_1 g_2^{-1}) = \phi(g_1)\phi(g_2^{-1}) = \phi(g_1)[\phi(g_2)]^{-1} \in B,$$

because B is a subgroup. Thus, $g_1 g_2^{-1} \in \phi^{-1}(B)$. Hence, $\phi^{-1}(B)$ is a subgroup of G. □

Theorem: Composition of Homomorphisms is a Homomorphism

Theorem 0.5. *Let* $\phi : G \to H$ *and* $\psi : H \to K$ *be group homomorphisms. Then the composition* $\psi \circ \phi : G \to K$ *defined by* $(\psi \circ \phi)(g) = \psi(\phi(g))$ *is also a group homomorphism.*

Proof. For any $a, b \in G$,

$$(\psi \circ \phi)(ab) = \psi(\phi(ab)) = \psi(\phi(a)\phi(b)) = \psi(\phi(a))\psi(\phi(b)) = (\psi \circ \phi)(a)(\psi \circ \phi)(b).$$

Hence, $\psi \circ \phi$ is a homomorphism. □

Corollary 0.1. *The composition of monomorphisms is a monomorphism. The composition of epimorphisms is an epimorphism. The composition of isomorphisms is an isomorphism.*

Theorem: Isomorphism Preserves Group Properties

Theorem 0.6. *Let* $\phi : G \to H$ *be an isomorphism. Then*

(i) G *is abelian if and only if* H *is abelian.*

(ii) G *is cyclic if and only if* H *is cyclic.*

(iii) For any $a \in G$, $|a| = |\phi(a)|$ *(order is preserved).*

(iv) G *is finite if and only if* H *is finite, and* $|G| = |H|$*.*

(v) The inverse mapping $\phi^{-1} : H \to G$ *is also an isomorphism.*

Proof. (i) Suppose G is abelian. For any $x, y \in H$, there exist $a, b \in G$ such that $\phi(a) = x$ and $\phi(b) = y$ (since ϕ is surjective). Then

$$xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx.$$

Thus, H is abelian. The converse follows similarly using ϕ^{-1}.

(ii) If $G = \langle a \rangle$ is cyclic, then any element of H is of the form $\phi(a^n) = \phi(a)^n$. Hence, $H = \langle \phi(a) \rangle$ is cyclic. The converse follows similarly.

(iii) Let $|a| = n$. Then $a^n = e_G$, so $\phi(a)^n = \phi(a^n) = \phi(e_G) = e_H$. Thus, $|\phi(a)|$ divides n. Similarly, applying ϕ^{-1}, n divides $|\phi(a)|$. Hence, $|a| = |\phi(a)|$.

(iv) This follows immediately from the fact that ϕ is bijective.

(v) Since ϕ is bijective, ϕ^{-1} exists. For any $x, y \in H$, let $\phi(a) = x$ and $\phi(b) = y$. Then

$$\phi^{-1}(xy) = \phi^{-1}(\phi(a)\phi(b)) = \phi^{-1}(\phi(ab)) = ab = \phi^{-1}(x)\phi^{-1}(y).$$

Thus, ϕ^{-1} is a homomorphism, hence an isomorphism. □

Theorem: Cayley's Theorem (Statement Only)

Theorem 0.7 (Cayley's Theorem)**.** *Every group G is isomorphic to a subgroup of the symmetric group S_G (the group of all permutations of the set G). In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.*

Proof. The proof is omitted here as it is covered in a separate section of the syllabus. It involves the left regular representation $L : G \to S_G$ defined by $L_g(x) = gx$. □

Theorem: First Isomorphism Theorem

Theorem 0.8 (First Isomorphism Theorem)**.** *If $\phi : G \to H$ is a group homomorphism, then $\ker \phi \lhd G$ and*

$$G/\ker \phi \cong Im(\phi).$$

Proof. From Theorem 0.2, $\ker \phi \lhd G$. Define a mapping $\overline{\phi} : G/\ker \phi \to \mathrm{Im}(\phi)$ by

$$\overline{\phi}(g \ker \phi) = \phi(g).$$

Well-defined: If $g_1 \ker \phi = g_2 \ker \phi$, then $g_1^{-1} g_2 \in \ker \phi$, so $\phi(g_1^{-1} g_2) = e_H$. Hence, $\phi(g_1) = \phi(g_2)$. Thus, $\overline{\phi}$ is well-defined.

Homomorphism: For any $g_1 \ker \phi, g_2 \ker \phi \in G/\ker \phi$,

$$\overline{\phi}((g_1 \ker \phi)(g_2 \ker \phi)) = \overline{\phi}(g_1 g_2 \ker \phi) = \phi(g_1 g_2) = \phi(g_1)\phi(g_2) = \overline{\phi}(g_1 \ker \phi)\overline{\phi}(g_2 \ker \phi).$$

Injectivity: Suppose $\overline{\phi}(g \ker \phi) = e_H$. Then $\phi(g) = e_H$, so $g \in \ker \phi$. Hence, $g \ker \phi = \ker \phi$, the identity in $G/\ker \phi$.

Surjectivity: By definition, every element of $\mathrm{Im}(\phi)$ is of the form $\phi(g)$ for some $g \in G$, and $\overline{\phi}(g \ker \phi) = \phi(g)$. Hence, $\overline{\phi}$ is surjective.

Therefore, $\overline{\phi}$ is an isomorphism, and $G/\ker \phi \cong \mathrm{Im}(\phi)$. □

Theorem: Second Isomorphism Theorem

Theorem 0.9 (Second Isomorphism Theorem)**.** *If H is a subgroup of G and N is a normal subgroup of G, then*

$$\frac{H}{H \cap N} \cong \frac{HN}{N}.$$

Proof. Define a mapping $\phi : H \to HN/N$ by $\phi(h) = hN$. Since $N \lhd G$, HN is a subgroup of G and $N \lhd HN$. The mapping ϕ is a homomorphism because

$$\phi(h_1h_2) = h_1h_2N = (h_1N)(h_2N) = \phi(h_1)\phi(h_2).$$

The kernel of ϕ is $\{h \in H : hN = N\} = \{h \in H : h \in N\} = H \cap N$.

The image of ϕ is $\{hN : h \in H\}$. For any $hnN \in HN/N$ (with $h \in H, n \in N$), we have $hnN = hN$ because $n \in N$. Thus, every element of HN/N is of the form hN for some $h \in H$. Hence, $\mathrm{Im}(\phi) = HN/N$.

By the First Isomorphism Theorem,

$$\frac{H}{\ker \phi} \cong \mathrm{Im}(\phi) \quad \Rightarrow \quad \frac{H}{H \cap N} \cong \frac{HN}{N}.$$

□

Theorem: Third Isomorphism Theorem

Theorem 0.10 (Third Isomorphism Theorem)**.** *If N and K are normal subgroups of G with $N \subseteq K \subseteq G$, then*

$$\frac{G/N}{K/N} \cong \frac{G}{K}.$$

Proof. Define a mapping $\phi : G/N \to G/K$ by $\phi(gN) = gK$. Since $N \subseteq K$, this mapping is well-defined: if $g_1N = g_2N$, then $g_1^{-1}g_2 \in N \subseteq K$, so $g_1K = g_2K$. ϕ is a homomorphism because

$$\phi((g_1N)(g_2N)) = \phi(g_1g_2N) = g_1g_2K = (g_1K)(g_2K) = \phi(g_1N)\phi(g_2N).$$

The kernel of ϕ is $\{gN \in G/N : \phi(gN) = K\} = \{gN \in G/N : g \in K\} = K/N$.

The image of ϕ is $\{gK : g \in G\} = G/K$, so ϕ is surjective. By the First Isomorphism Theorem,

$$\frac{G/N}{\ker \phi} \cong \mathrm{Im}(\phi) \quad \Rightarrow \quad \frac{G/N}{K/N} \cong \frac{G}{K}.$$

□

Step-by-Step Procedure to Find the Kernel and Image

To find the kernel and image of a homomorphism $\phi : G \to H$, the following steps may be followed:

Step 1. **Kernel:** Solve $\phi(g) = e_H$ for $g \in G$. The set of all solutions is $\ker \phi$.

Step 2. **Image:** Compute $\phi(g)$ for all $g \in G$ (or for generators of G). The set of all such values is $\mathrm{Im}(\phi)$.

Step 3. **Verify:** Check that $\ker \phi \lhd G$ and that $\mathrm{Im}(\phi) \leq H$.

Common Mistakes to Avoid

The following are frequent errors encountered while studying homomorphisms:

- **Forgetting to check that $\phi(ab) = \phi(a)\phi(b)$ for *all* a, b:** A single counterexample suffices to show that a mapping is not a homomorphism.
- **Assuming that $\phi(a^{-1}) = [\phi(a)]^{-1}$ is automatically true:** This is a property that must be proved, not assumed.
- **Confusing kernel with image:** The kernel is a subset of the domain; the image is a subset of the codomain.
- **Thinking that every homomorphism is either injective or surjective:** A homomorphism can be neither, one, or both.
- **Assuming that the image of a normal subgroup is always normal:** The image of a normal subgroup under a homomorphism is normal in the image of the domain, but not necessarily in the whole codomain.

Practice Problems

Q1. Define $\phi : \mathbb{Z} \to \mathbb{Z}_n$ by $\phi(m) = m \mod n$. Show that ϕ is a homomorphism. Find its kernel and image.

Q2. Define $\phi : \mathbb{R}^* \to \mathbb{R}^*$ by $\phi(x) = |x|$. Show that ϕ is a homomorphism. Find its kernel and image.

Q3. Let $\phi : G \to H$ be a homomorphism. Prove that if $a \in G$ has finite order n, then the order of $\phi(a)$ divides n.

Q4. Let $\phi : G \to H$ be a homomorphism. Prove that ϕ is injective if and only if $\ker \phi = \{e_G\}$.

Q5. Let $\phi : G \to H$ be a homomorphism. Prove that for any subgroup A of G, $\phi(A)$ is a subgroup of H.

Q6. Let $\phi : G \to H$ be a homomorphism. Prove that if B is a subgroup of H, then $\phi^{-1}(B)$ is a subgroup of G.

Q7. Prove that the composition of two homomorphisms is a homomorphism.

Q8. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(n) = 2n$. Is ϕ a homomorphism? Is it injective? Is it surjective?

Q9. Let $G = GL(2, \mathbb{R})$ and $H = \mathbb{R}^*$. Define $\phi(A) = \det(A)$. Find $\ker \phi$ and $\text{Im}(\phi)$.

Q10. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.

Summary Table for Quick Revision

Type of Homomorphism	Condition
Monomorphism (injective)	$\ker\phi = \{e_G\}$
Epimorphism (surjective)	$\mathrm{Im}(\phi) = H$
Isomorphism (bijective)	$\ker\phi = \{e_G\}$ and $\mathrm{Im}(\phi) = H$
Endomorphism	$\phi : G \to G$
Automorphism	$\phi : G \to G$ bijective

Property	Result
$\phi(e_G)$	$= e_H$
$\phi(a^{-1})$	$= [\phi(a)]^{-1}$
$\phi(a^n)$	$= [\phi(a)]^n$
$\|a\|$ finite	$\|\phi(a)\|$ divides $\|a\|$
$\ker\phi$	$\lhd G$
$\mathrm{Im}(\phi)$	$\leq H$

Exam Preparation Checklist

Before appearing for the university examination, the student should ensure the following:

- Can state the definition of a group homomorphism.
- Can provide examples of homomorphisms (trivial, identity, inclusion, exponential, determinant, sign).
- Can state and prove the basic properties of homomorphisms (preservation of identity, inverses, powers).
- Can prove that the kernel is a normal subgroup.
- Can prove that a homomorphism is injective if and only if the kernel is trivial.
- Can prove that the image of a subgroup is a subgroup.
- Can prove that the composition of homomorphisms is a homomorphism.
- Can state and prove the First, Second, and Third Isomorphism Theorems.
- Can apply the Isomorphism Theorems to solve problems.
