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Department of Mathematics

Class Note

B.Sc. Mathematics (Major)
Semester IV
Course Code: MATH-M-T-05
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Unit 1: Normal Subgroups

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Introduction

For any subgroup H of a group G, the left cosets gH and the right cosets Hg can be formed. In general, these two sets are not necessarily equal. Subgroups for which left cosets coincide with right cosets are of particular importance because they allow the construction of quotient groups. Such subgroups are called **normal subgroups**.

Key Concept: A normal subgroup is a subgroup that is invariant under conjugation by any element of the group. Normal subgroups are precisely those subgroups that can serve as kernels of homomorphisms and are used to form factor groups.

Definition of Normal Subgroup

Definition 0.1. *A subgroup N of a group G is called a* ***normal subgroup*** *if for every $g \in G$ and every $n \in N$, the element gng^{-1} belongs to N. Symbolically,*

$$gNg^{-1} \subseteq N \quad \text{for all } g \in G.$$

The notation $N \triangleleft G$ is used to denote that N is a normal subgroup of G.

Theorem 0.1 (Equivalent Conditions for Normality)**.** *For a subgroup N of a group G, the following statements are equivalent:*

(1) $gng^{-1} \in N$ *for all* $g \in G$ *and all* $n \in N$.

(2) $gNg^{-1} \subseteq N$ *for all* $g \in G$.

(3) $gNg^{-1} = N$ *for all* $g \in G$.

(4) $gN = Ng$ *for all* $g \in G$ *(i.e., left cosets equal right cosets).*

Proof. (1) ⇒ (2): This is immediate from the definition of subset inclusion.

(2) ⇒ (3): From (2), we have $gNg^{-1} \subseteq N$ for all $g \in G$. Replacing g by g^{-1}, we get $g^{-1}Ng \subseteq N$. Multiplying on the left by g and on the right by g^{-1}, we obtain $N \subseteq gNg^{-1}$. Hence, $gNg^{-1} = N$.

(3) ⇒ (4): If $gNg^{-1} = N$, then multiplying on the right by g, we get $gN = Ng$.

(4) ⇒ (1): If $gN = Ng$, then for any $n \in N$, there exists some $n' \in N$ such that $gn = n'g$. Then $gng^{-1} = n' \in N$.

Thus all four statements are equivalent. □

Examples of Normal Subgroups

Example 0.1 (Trivial Normal Subgroups)**.** *In any group G, the subgroups $\{e\}$ and G itself are always normal. These are called the* ***trivial normal subgroups****.*

Example 0.2 (Abelian Groups)**.** *If G is an abelian group, then every subgroup H of G is normal. This holds because for any $g \in G$ and $h \in H$, $ghg^{-1} = h \in H$.*

Example 0.3 (The Alternating Group)**.** *The alternating group A_n is a normal subgroup of the symmetric group S_n for all $n \geq 2$. The index $[S_n : A_n] = 2$, and Theorem 0.2 proves that any subgroup of index 2 is normal.*

Example 0.4 (The Center of a Group)**.** *The center $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$ is always a normal subgroup of G. For any $g \in G$ and $z \in Z(G)$, $gzg^{-1} = z \in Z(G)$.*

Example 0.5 (Normal Subgroups of S_3)**.** *The symmetric group S_3 has the following normal subgroups:*

- $\{e\}$
- $A_3 = \{e, (123), (132)\}$ *(since* $[S_3 : A_3] = 2$*)*
- S_3 *itself.*

The subgroup $\{e, (12)\}$ is not normal in S_3.

Non-Examples (Subgroups that are NOT Normal)

Example 0.6. *Consider $G = S_3$, the symmetric group on three symbols. Let $H = \{e, (12)\}$. Take $g = (13)$ and $n = (12)$. Then*

$$gng^{-1} = (13)(12)(13)^{-1} = (13)(12)(13) = (23).$$

Since $(23) \notin H$, it follows that H is ***not*** *normal in S_3.*

Important: In non-abelian groups, normality is a special property. A subgroup must be invariant under conjugation by *every* element of the group to be normal.

Theorem: Subgroups of Index 2 are Normal

Theorem 0.2. *If H is a subgroup of a group G such that the index $[G : H] = 2$ (i.e., H has exactly two left cosets in G), then H is normal in G.*

Proof. Since $[G : H] = 2$, there are exactly two left cosets. One of them is H itself. Let $g \in G$ be any element not belonging to H. Then the left cosets are H and gH. Similarly, the right cosets are H and Hg. Since the left cosets partition G and the right cosets also partition G, we have

$$G = H \cup gH \quad \text{and} \quad G = H \cup Hg,$$

where the unions are disjoint. Since $g \notin H$, we have $g \in gH$ and $g \in Hg$. The only coset containing g on the left is gH, and the only coset containing g on the right is Hg. Therefore, $gH = Hg$. For any $h \in H$, trivially $hH = H = Hh$. Hence, for every $x \in G$, we have $xH = Hx$. Thus, by Theorem 0.1, $H \triangleleft G$. □

Corollary 0.1. *The alternating group A_n is normal in S_n for all $n \geq 2$ because $[S_n : A_n] = 2$.*

Theorem: Intersection of Normal Subgroups is Normal

Theorem 0.3. *If N and K are normal subgroups of a group G, then $N \cap K$ is also a normal subgroup of G.*

Proof. It is known that the intersection of subgroups is a subgroup. To prove normality, let $g \in G$ and $x \in N \cap K$. Then $x \in N$ and $x \in K$. Since $N \lhd G$, $gxg^{-1} \in N$. Since $K \lhd G$, $gxg^{-1} \in K$. Therefore, $gxg^{-1} \in N \cap K$. Hence, $N \cap K \lhd G$. □

Corollary 0.2. *The intersection of any collection of normal subgroups of a group G is normal in G.*

Theorem: Product of Normal Subgroups is Normal

Theorem 0.4. *If N and K are normal subgroups of a group G, then the product*

$$NK = \{nk : n \in N, k \in K\}$$

is a normal subgroup of G.

Proof. Since N and K are normal, they are subgroups. It is known that if at least one of N or K is normal, then NK is a subgroup of G. To prove normality, let $g \in G$ and $nk \in NK$. Then

$$g(nk)g^{-1} = (gng^{-1})(gkg^{-1}) \in NK,$$

because $gng^{-1} \in N$ and $gkg^{-1} \in K$ by normality. Thus, $g(NK)g^{-1} \subseteq NK$ for all $g \in G$. Hence, $NK \lhd G$. □

Theorem: If $N \lhd G$ and $H \subset G$, then $N \cap H \lhd H$

Theorem 0.5. *If N is a normal subgroup of G and H is any subgroup of G, then $N \cap H$ is a normal subgroup of H.*

Proof. It is known that $N \cap H$ is a subgroup of H. Let $x \in H$ and $y \in N \cap H$. Then $x \in G$ and $y \in N$. Since $N \lhd G$, we have $xyx^{-1} \in N$. Also, because $x, y \in H$ and H is a subgroup, $xyx^{-1} \in H$. Hence, $xyx^{-1} \in N \cap H$. Therefore, $N \cap H \lhd H$. □

Theorem: If $N \lhd G$ and $H \subset G$, then HN is a subgroup

Theorem 0.6. *If N is a normal subgroup of G and H is any subgroup of G, then*

$$HN = \{hn : h \in H, n \in N\}$$

is a subgroup of G. Moreover, $N \lhd HN$.

Proof. Since $e = ee \in HN$, HN is non-empty. Let $h_1n_1, h_2n_2 \in HN$. Then

$$(h_1n_1)(h_2n_2)^{-1} = h_1n_1n_2^{-1}h_2^{-1} = h_1(h_2^{-1}h_2)n_1n_2^{-1}h_2^{-1} = (h_1h_2^{-1})(h_2n_1n_2^{-1}h_2^{-1}).$$

Since $N \lhd G$, $h_2 n_1 n_2^{-1} h_2^{-1} \in N$. Also, $h_1 h_2^{-1} \in H$. Hence, the product belongs to HN. Thus, HN is a subgroup of G. To prove $N \lhd HN$, note that $N \subseteq HN$ and for any $hn \in HN$ and any $n_1 \in N$,

$$(hn)n_1(hn)^{-1} = h(nn_1n^{-1})h^{-1} \in N,$$

because $nn_1n^{-1} \in N$ and conjugation by h preserves N. Hence, $N \lhd HN$. □

Theorem: The Normalizer is a Subgroup

Theorem 0.7. *Let H be a subgroup of a group G. Then the normalizer*

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

is a subgroup of G containing H. Moreover, $H \lhd N_G(H)$.

Proof. Clearly, $e \in N_G(H)$ because $eHe^{-1} = H$. Let $g_1, g_2 \in N_G(H)$. Then

$$(g_1g_2)H(g_1g_2)^{-1} = g_1(g_2Hg_2^{-1})g_1^{-1} = g_1Hg_1^{-1} = H,$$

so $g_1g_2 \in N_G(H)$. Also,

$$g_1^{-1}Hg_1 = g_1^{-1}(g_1Hg_1^{-1})g_1 = H,$$

so $g_1^{-1} \in N_G(H)$. Hence, $N_G(H)$ is a subgroup of G. For any $h \in H$, $hHh^{-1} = H$, so $h \in N_G(H)$. Thus, $H \subseteq N_G(H)$. By definition of $N_G(H)$, for every $g \in N_G(H)$, $gHg^{-1} = H$, which means $H \lhd N_G(H)$. □

Corollary 0.3. *$H \lhd G$ if and only if $N_G(H) = G$.*

Theorem: The Center is a Normal Subgroup

Theorem 0.8. *The center $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$ is a normal subgroup of G.*

Proof. It is known that $Z(G)$ is a subgroup of G. Let $z \in Z(G)$ and $g \in G$. Then

$$gzg^{-1} = zgg^{-1} = z \in Z(G).$$

Thus, $gZ(G)g^{-1} \subseteq Z(G)$ for all $g \in G$. Hence, $Z(G) \lhd G$. □

Step-by-Step Procedure to Check Normality

To determine whether a given subgroup H of a group G is normal, the following steps may be followed:

Step 1. If G is abelian, then H is automatically normal (no calculation needed).

Step 2. If $[G : H] = 2$, then H is automatically normal (by Theorem 0.2).

Step 3. Otherwise, take an arbitrary $g \in G$ and an arbitrary $h \in H$.

Step 4. Compute ghg^{-1}.

Step 5. Check whether $ghg^{-1} \in H$.

Step 6. If this holds for all choices, then H is normal. If a single counterexample is found, then H is not normal.

Common Mistakes to Avoid

The following are frequent errors encountered while studying normal subgroups:

- **Assuming every subgroup of a non-abelian group is normal:** This is false. The example $H = \{(1), (12)\}$ in S_3 demonstrates that not all subgroups are normal.
- **Confusing normalizer with centralizer:** The normalizer of a subgroup is the set of elements that conjugate the subgroup to itself. The centralizer of an element is the set of elements that commute with that element.
- **Forgetting to check all $g \in G$:** A single counterexample suffices to show non-normality. However, to prove normality, the condition must be verified for every $g \in G$.
- **Misinterpreting $gN = Ng$:** The equality $gN = Ng$ means the sets are equal, not that every element of N commutes with g.

Practice Problems

Q1. Let $G = GL(2, \mathbb{R})$ (the group of invertible 2×2 real matrices). Show that the subgroup $SL(2, \mathbb{R})$ (matrices with determinant 1) is normal in G.

Q2. Let $G = D_4$ (the dihedral group of order 8). Find all normal subgroups of G.

Q3. Prove that the center $Z(G)$ is always a normal subgroup of G.

Q4. Let H be a subgroup of G such that every left coset of H is also a right coset of H. Prove that H is normal in G.

Q5. Give an example of a group G and a subgroup H such that H is not normal, but $N_G(H) \neq H$.

Q6. Find all normal subgroups of the quaternion group Q_8.

Q7. Prove that if $N \triangleleft G$ and $K \triangleleft G$ with $N \cap K = \{e\}$, then $nk = kn$ for all $n \in N$, $k \in K$.

Summary Table for Quick Revision

Condition	Conclusion
G is abelian	Every subgroup is normal
$[G : H] = 2$	$H \triangleleft G$
$H = Z(G)$	$H \triangleleft G$
$H = G$ or $H = \{e\}$	$H \triangleleft G$
$gHg^{-1} = H$ for all $g \in G$	$H \triangleleft G$
$H = \ker \phi$ for some homomorphism ϕ	$H \triangleleft G$
H is a union of conjugacy classes	$H \triangleleft G$

Exam Preparation Checklist

Before appearing for the university examination, the student should ensure the following:

- Can state the definition of a normal subgroup in multiple equivalent ways.
- Can check whether a given subgroup is normal.
- Can provide examples of normal subgroups and non-normal subgroups.
- Understands why subgroups of index 2 are always normal.
- Can distinguish between center, centralizer, and normalizer.
- Can prove that the kernel of a homomorphism is always a normal subgroup.
- Can prove that the center is a normal subgroup.
- Can prove that the intersection and product of normal subgroups are normal.
