DUMKAL COLLEGE

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Department of Mathematics

Class Note

B.Sc. Mathematics (Major)
Semester VI
Course Code: MATH-M-T-09
Course Title: Linear Programming Problems & Game Theory

Unit 1: Graphical Solution of LPP

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CLASS NOTE – Concept 2

(Graphical Method for 2-Variable LPPs)

When to Use the Graphical Method?

The graphical method is used to solve Linear Programming Problems that have **exactly two decision variables**. The reason is simple: we can plot the constraints on a two-dimensional xy-plane. If there are three or more variables, we cannot draw the feasible region, and we must use the Simplex method instead.

> **Important:** Graphical method works only for LPPs with 2 variables. For 3 or more variables, we use the Simplex method.

Step-by-Step Procedure for Graphical Solution

I follow these seven steps every time I solve an LPP graphically:

Step 1. **Formulate the LPP** (if not already given). Write the objective function and all constraints.

Step 2. **Convert inequalities to equalities.** Replace $\leq$ or $\geq$ with $=$ to get equations of straight lines.

Step 3. **Plot each line** on graph paper. Find two points for each line (usually the intercepts).

Step 4. **Identify the feasible region.** This is the common area that satisfies **all** constraints simultaneously. Shade it clearly.

Step 5. **Find all corner points** of the feasible region. These are the intersection points of the boundary lines.

Step 6. **Evaluate the objective function** at each corner point.

Step 7. **Select the optimal solution.** For a maximization problem, choose the corner point with the largest Z value. For minimization, choose the smallest Z value.

Important Rules for Shading

When plotting constraints, the direction of shading depends on the inequality sign:

Inequality	Shade the side where...
$ax + by \leq c$	The origin (0,0) satisfies the inequality
$ax + by \geq c$	The origin (0,0) does NOT satisfy the inequality
$x \geq 0$	Right side of the y-axis
$y \geq 0$	Above the x-axis

Test point method: Pick a test point (usually the origin). If it satisfies the inequality, shade the side containing the origin. Otherwise, shade the opposite side.

Example 1: Maximization Problem (Solved Step by Step)

Problem: Solve the following LPP graphically.

$$\begin{aligned} \text{Maximize } Z &= 3x + 5y \\ \text{subject to} \quad x + y &\leq 6 \\ x &\leq 4 \\ y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

Solution (as I write in my notebook):

Step 1: The LPP is already formulated.

Step 2: Convert inequalities to equalities:

$$x + y = 6, \quad x = 4, \quad y = 5, \quad x = 0, \quad y = 0$$

Step 3: Find intercepts for each line:

- $x + y = 6$: passes through $(6, 0)$ and $(0, 6)$
- $x = 4$: vertical line through $(4, 0)$ and $(4, 5)$
- $y = 5$: horizontal line through $(0, 5)$ and $(4, 5)$
- $x = 0$: the y-axis
- $y = 0$: the x-axis

Step 4: The feasible region is shown in Figure 1 below.

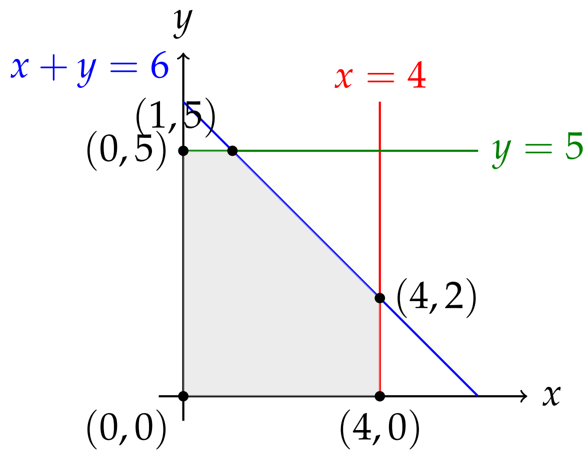

Figure 1: Feasible Region (Maximization)

The shaded polygon $O(0,0) \to (4,0) \to (4,2) \to (1,5) \to (0,5)$ is the feasible region.

Step 5: Find the corner points by solving pairs of equations:

Intersection of	Corner point
$x = 0,\ y = 0$	$(0,0)$
$y = 0,\ x = 4$	$(4,0)$
$x = 4,\ x + y = 6$	$(4,2)$
$x + y = 6,\ y = 5$	$(1,5)$
$x = 0,\ y = 5$	$(0,5)$

Step 6: Evaluate $Z = 3x + 5y$ at each corner point:

Corner point	$Z = 3x + 5y$
$(0,0)$	0
$(4,0)$	12
$(4,2)$	$3(4) + 5(2) = 12 + 10 = 22$
$(1,5)$	$3(1) + 5(5) = 3 + 25 = 28$
$(0,5)$	25

Step 7: The maximum value of Z is **28**, which occurs at the corner point $(1,5)$.

$$\text{Optimal solution: } x = 1,\ y = 5,\ Z_{\max} = 28$$

Example 2: Minimization Problem

Problem: Solve the following LPP graphically.

$$\begin{aligned} \text{Minimize } Z &= 4x + 3y \\ \text{subject to} \quad x + y &\geq 3 \\ x &\geq 1 \\ y &\geq 1 \\ x, y &\geq 0 \end{aligned}$$

Solution:

Step 1-2: Plot the lines $x + y = 3$, $x = 1$, $y = 1$.

Step 3: The feasible region is shown in Figure 2 below.

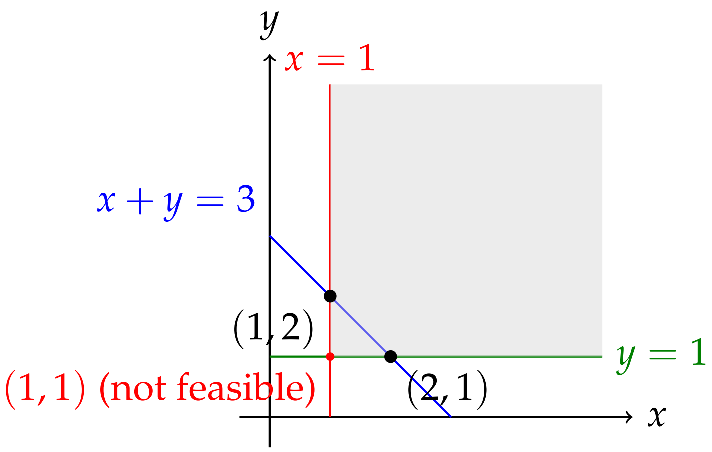

Figure 2: Feasible Region (Minimization)

The feasible region is unbounded (it extends infinitely to the top-right). The corner points are $(1, 2)$ and $(2, 1)$. Note that $(1, 1)$ is not in the feasible region because $1 + 1 \geq 3$ is false.

Step 4: Evaluate $Z = 4x + 3y$:

Point	Z
$(1, 2)$	$4(1) + 3(2) = 4 + 6 = 10$
$(2, 1)$	$4(2) + 3(1) = 8 + 3 = 11$

Step 5: The minimum value is **10** at $(1, 2)$.

$$\boxed{\text{Optimal solution: } x = 1,\ y = 2,\ Z_{\min} = 10}$$

Special Cases in Graphical Solution

Sometimes we encounter special situations. I must recognize them:

- **No feasible solution:** The constraints are contradictory. The feasible region is empty.
- **Unbounded solution:** The feasible region extends infinitely in some direction, and the objective function can increase (or decrease) without bound. This usually happens in maximization problems when the region is not closed.
- **Multiple optimal solutions:** The objective function line is parallel to one of the constraint boundaries. In this case, every point on that edge gives the same optimal Z value.
- **Redundant constraint:** A constraint that does not affect the feasible region because it is automatically satisfied by the other constraints.

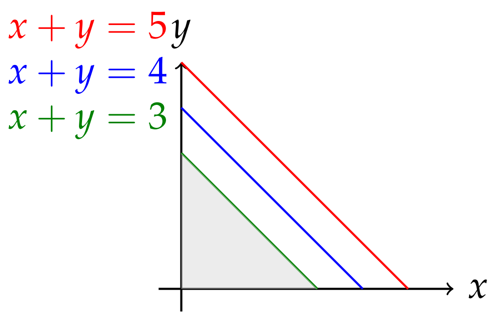

Figure 3: Multiple Optimal Solutions

> **Remember:** For a maximization problem, if the feasible region is unbounded, the solution may be unbounded (no finite maximum). For minimization, an unbounded region can still have a finite minimum.

Common Mistakes (I have made these before!)

- **Shading the wrong side of a constraint:** Always use the origin test (unless the origin lies on the line, then use another test point like $(1,0)$ or $(0,1)$).

- **Forgetting the non-negativity constraints:** The feasible region must be in the first quadrant unless the problem says otherwise.
- **Missing a corner point:** Some corner points come from intersection of two lines that are not both constraints. Check all pairs of boundary lines.
- **Misreading the optimal value:** For minimization, the smallest Z is optimal, not the largest.
- **Not drawing the graph to scale:** Uneven scaling can lead to wrong corner points.

Quick Reference Table

Type of problem	Optimal solution is at...
Maximization	Corner point with largest Z
Minimization	Corner point with smallest Z

Inequality sign	Shading direction (origin test)
$\leq$	Towards origin
$\geq$	Away from origin

Practice Problems (Do in notebook)

Q1. Solve graphically:

$$\begin{aligned} \text{Maximize } Z &= 2x + 3y \\ \text{subject to } x + 2y &\leq 8 \\ 2x + y &\leq 10 \\ x, y &\geq 0 \end{aligned}$$

Q2. Solve graphically:

$$\begin{aligned} \text{Minimize } Z &= 3x + 2y \\ \text{subject to } 2x + y &\geq 6 \\ x + y &\geq 4 \\ x, y &\geq 0 \end{aligned}$$

Q3. Identify the special case:

$$\begin{aligned} \text{Maximize } Z &= x + y \\ \text{subject to } x - y &\leq 1 \\ x + y &\geq 3 \\ x, y &\geq 0 \end{aligned}$$

Exit Check (Ask yourself before next class)

- Can I plot a linear inequality correctly?
- Do I know how to find the intersection points of two lines?
- Can I identify the feasible region from a graph?
- Do I understand why the optimal solution is always at a corner point?
- Can I recognize special cases like no solution or multiple solutions?

Next topic: Convex Sets and Basic Solutions.