

Ordered Sets and Lattices

Complete Final Examination Notes

Unit 2: Ordered Sets and Lattices

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These notes are prepared exclusively for University Examination preparation.

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1 Ordered Sets (Partially Ordered Sets)

Definition 1: Partial Order (Poset)

A relation R on a set S is a **partial order** if it satisfies:

1. **Reflexive:** For any $a \in S$, aRa .
2. **Antisymmetric:** If aRb and bRa , then $a = b$.
3. **Transitive:** If aRb and bRc , then aRc .

The set S with a partial order R is called a **partially ordered set** or **poset**, denoted (S, R) .

Standard Notation

A partial order is usually denoted by $\preceq$ (read “precedes”).

- $a \preceq b$: a precedes b .
- $a \prec b$: $a \preceq b$ and $a \neq b$ (strictly precedes).
- $a \succeq b$: a succeeds b ($b \preceq a$).
- $a \succ b$: a strictly succeeds b ($b \prec a$).
- $a \parallel b$: a and b are **noncomparable** (neither $a \preceq b$ nor $b \preceq a$).

When unambiguous, $\leq, <, \geq, >$ are used instead of $\preceq, \prec, \succeq, \succ$.

Example 1 (Example 1: Set Inclusion). *Let S be any collection of sets. The relation $\subseteq$ is a partial order.*

- *Reflexive:* $A \subseteq A$.
- *Antisymmetric:* $A \subseteq B$ and $B \subseteq A \implies A = B$.
- *Transitive:* $A \subseteq B$ and $B \subseteq C \implies A \subseteq C$.

Example 2 (Example 2: Divisibility on $\mathbb{N}$). *Let $\mathbb{N}$ be positive integers. “ a divides b ” ($a \mid b$) is a partial order.*

- *Reflexive:* $a \mid a$ (since $a \cdot 1 = a$).
- *Antisymmetric:* $a \mid b$ and $b \mid a \implies a = b$.
- *Transitive:* $a \mid b$ and $b \mid c \implies a \mid c$.

Remark 3 (Remark 1). *Divisibility is **not** a partial order on $\mathbb{Z}$ (integers). It fails antisymmetry: $2 \mid -2$ and $-2 \mid 2$, but $2 \neq -2$.*

1.1 Dual Order and Subsets

Definition 2: Dual Order

If $\preceq$ is a partial order on S , the relation $\succeq$ (succeeds) is also a partial order, called the **dual order**. It is the inverse relation: $\succeq = \preceq^{-1}$.

Definition 3: Ordered Subset

If $A \subseteq S$ and S is a poset, the **induced order** on A is defined by $a \preceq b$ in A if and only if $a \preceq b$ in S .

1.2 Quasi-Order**Definition 4: Quasi-Order**

A relation $\prec$ on S is a **quasi-order** if:

1. **Irreflexive:** $a \not\prec a$ for any a .
2. **Transitive:** $a \prec b$ and $b \prec c \implies a \prec c$.

Conversion Between Order and Quasi-Order

- If $\preceq$ is a partial order, define $\prec$ by $a \prec b \iff a \preceq b$ and $a \neq b$. Then $\prec$ is a quasi-order.
- If $\prec$ is a quasi-order, define $\preceq$ by $a \preceq b \iff a \prec b$ or $a = b$. Then $\preceq$ is a partial order.

1.3 Comparability and Linear Order**Definition 5: Comparability and Linear Order**

- **Comparable:** a and b are comparable if $a \preceq b$ or $b \preceq a$.
- **Noncomparable:** Written $a \parallel b$.
- **Totally/Linearly Ordered (Chain):** If **every** pair of elements in S is comparable, S is a chain.

Example 4 (Example 3: Linear vs Partial). • $\mathbb{N}$ with usual $\leq$ is a chain (linearly ordered).

- $\mathbb{N}$ with divisibility is **not** a chain (e.g., $3 \parallel 5$ since neither $3 \mid 5$ nor $5 \mid 3$).
- $A = \{2, 6, 12, 36\}$ is a chain under divisibility since $2 \mid 6, 6 \mid 12, 12 \mid 36$.
- $\mathcal{P}(X)$ (power set) with $\subseteq$ is not a chain if $|X| \geq 2$.

1.4 Product Sets and Order**Definition 6: Product Order**

For posets S and T , the **product order** on $S \times T$ is:

$$(a, b) \preceq (a', b') \iff a \preceq a' \text{ in } S \text{ and } b \preceq b' \text{ in } T$$

Definition 7: Lexicographical (Dictionary) Order

For **linearly** ordered sets S and T :

$$(a, b) \prec (a', b') \iff a < a' \text{ or } (a = a' \text{ and } b < b')$$

This extends to $S_1 \times \cdots \times S_n$ by comparing the first differing coordinate. This order is

linear.

1.5 Kleene Closure and Order

Let A be a linearly ordered alphabet, A^* the set of all words.

Definition 8: Alphabetical Order

1. $\lambda < w$ (empty word is first).
2. $u = au' \prec v = bv' \iff a < b$ or $(a = b \text{ and } u' \prec v')$.

Definition 9: Short-Lex Order (Free Semigroup Order)

$u \prec v \iff |u| < |v|$ or $(|u| = |v| \text{ and } u \text{ precedes } v \text{ alphabetically})$.

2 Hasse Diagrams of Partially Ordered Sets

Definition 10: Cover (Immediate Predecessor/Successor)

a is an **immediate predecessor** of b (or b **covers** a), if $a \prec b$ and **no element** exists in S such that $a \prec c \prec b$.

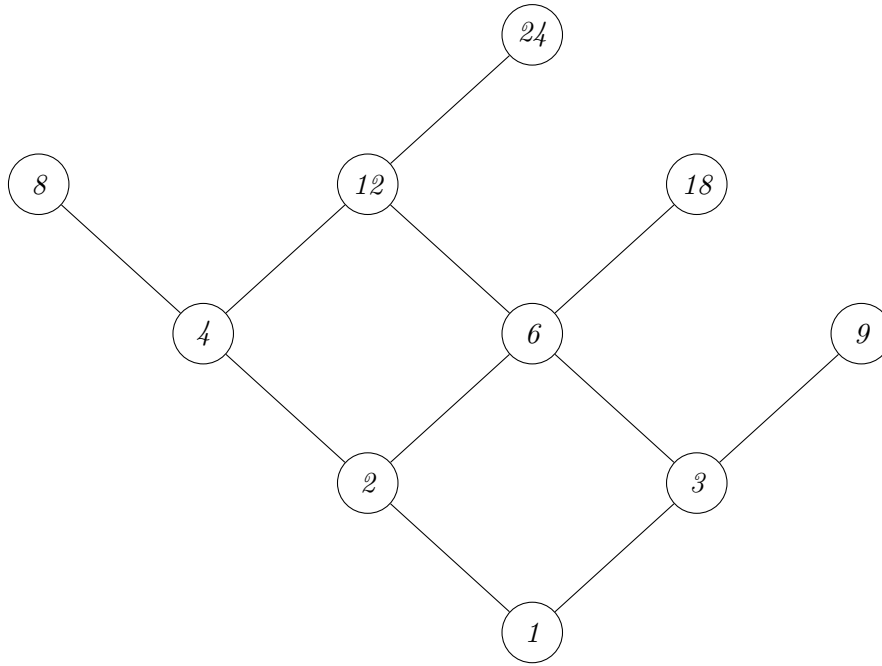
Hasse Diagram Construction

A Hasse diagram of a finite poset S is a directed graph where:

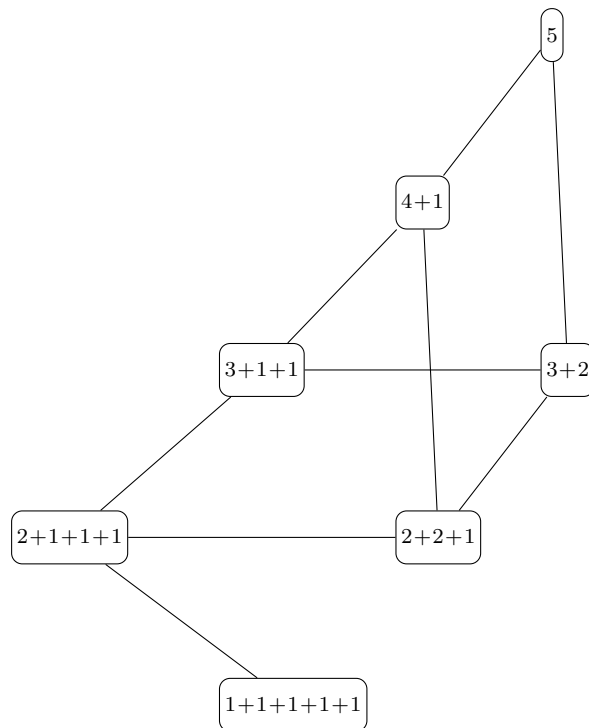
- Vertices are elements of S .
- An edge goes upward from a to b if a is an immediate predecessor of b .
- Transitive edges are omitted.
- Reflexive loops are omitted.

Key Property: $x \prec y$ if and only if x is an immediate predecessor of y or there exists a path of immediate predecessors from x to y .

Example 5 (Example 4: Divisibility Hasse Diagram). $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ ordered by “divides”.



Example 6 (Example 5: Partitions of 5). A partition of 5 is a set of positive integers summing to 5. Order: $P_1 \preceq P_2$ if parts of P_1 can be added to get parts of P_2 .



- $3 + 1 + 1$ and $2 + 2 + 1$ are **noncomparable**.
- Top element: 5. Bottom: $1 + 1 + 1 + 1 + 1$.

2.1 Minimal, Maximal, First, and Last Elements

Definition 11: Extremal Elements

- **Minimal Element** (a): No element strictly precedes a .
- **Maximal Element** (b): No element strictly succeeds b .
- **First Element** (a): $a \preceq x$ for **all** $x \in S$. (Unique if exists; must be the only minimal).
- **Last Element** (b): $x \preceq b$ for **all** $x \in S$. (Unique if exists; must be the only maximal).

Remark 7 (Remark 2: Existence Rules). • *Finite sets **always** have at least one minimal and one maximal element.*

- *Infinite sets may have none (e.g., $\mathbb{Z}$ with $\leq$).*
- *First/Last elements may not exist even in finite sets.*

Example 8 (Example 6: Analyzing Extremal Elements). 1. **Divisibility Set A (Example 4)**: Maximal elements are 18, 24 (neither is last). Minimal element is 1 (which is also First).

2. **Power Set $\mathcal{P}(X)$** : $\emptyset$ is First. X is Last.

3 Consistent Enumeration

Definition 12: Consistent Enumeration

A function $f : S \rightarrow \mathbb{N}$ is a **consistent enumeration** of a poset S if

$$a \prec b \implies f(a) < f(b)$$

Theorem 9 (Theorem 1: Existence of Consistent Enumeration). *There exists a consistent enumeration for any **finite** poset S . If S has n elements, there exists a mapping $f : S \rightarrow \{1, 2, \dots, n\}$.*

Remark 10 (Remark 3: Uniqueness). *Consistent enumerations are generally **not unique**. If $d \parallel e$ (noncomparable), we can assign $f(d) = 1, f(e) = 2$ or $f(d) = 2, f(e) = 1$. For a chain (linear order), the enumeration is unique.*

4 Supremum and Infimum

Definition 13: Bounds

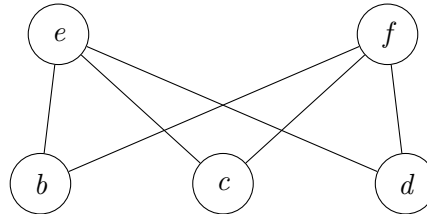
Let A be a subset of a poset S .

- **Upper Bound** (M): $x \preceq M$ for all $x \in A$.
- **Lower Bound** (m): $m \preceq x$ for all $x \in A$.
- **Bounded Above/Below**: If an upper/lower bound exists.
- **Bounded**: If it has both.

Definition 14: Supremum and Infimum

- **Supremum (Sup / Least Upper Bound – lub):** An upper bound M of A such that $M \preceq M'$ for every upper bound M' . Written $\sup(A)$ or $\text{lub}(A)$.
 - **Infimum (Inf / Greatest Lower Bound – glb):** A lower bound m of A such that $m' \preceq m$ for every lower bound m' . Written $\inf(A)$ or $\text{glb}(A)$.
- $\sup(A)$ and $\inf(A)$ are unique if they exist.

Example 11 (Example 7: Non-existence of Sup). Let $S = \{a, b, c, d, e, f\}$, $A = \{b, c, d\}$. Suppose e, f are upper bounds of A but $e \parallel f$.



- Upper bounds: $\{e, f\}$.
- Since e and f are noncomparable, neither precedes the other.
- Therefore, $\sup(A)$ **does not exist**.

Example 12 (Example 8: Existence of Sup and Inf). Let $S = \{1, \dots, 8\}$ ordered by divisibility. Let $A = \{4, 5, 7\}$.

- Upper bounds (numbers dividing all of 4, 5, 7): $\{1\}$.
- $\sup(A) = 1$.
- Lower bounds (multiples of all of 4, 5, 7): The LCM is 140. If $140 \notin S$, then $\inf(A)$ does not exist.

4.1 Supremum and Infimum in Divisibility Lattices

Crucial Connection: Number Theory and Order Theory

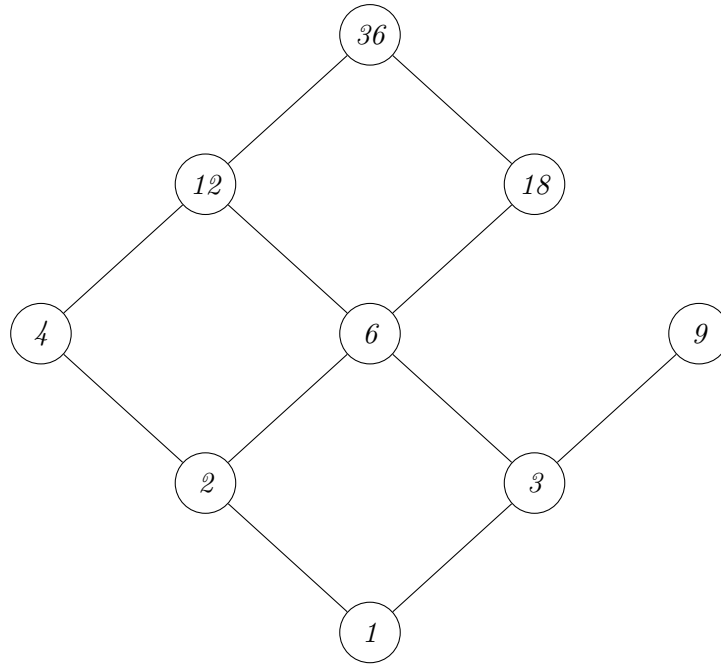
In the poset $(\mathbb{N}, |)$ (Positive integers ordered by divisibility):

- $\inf(a, b) = \gcd(a, b)$ (Greatest Common Divisor)
- $\sup(a, b) = \text{lcm}(a, b)$ (Least Common Multiple)

Reason:

- $\gcd$ is the “largest” number dividing both $\implies$ it is the “greatest lower bound” (infimum).
- lcm is the “smallest” number both divide $\implies$ it is the “least upper bound” (supremum).

Example 13 (Example 9: The Divisor Lattice D_m). Let D_m be the set of divisors of m ordered by divisibility. For $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$:



- $\inf(4, 6) = \gcd(4, 6) = 2$.
- $\sup(4, 6) = \text{lcm}(4, 6) = 12$.
- Every pair of elements in D_m has a sup and inf.

5 Lattices as Ordered Sets

Definition 15: Lattice

A poset $(L, \preceq)$ is called a **lattice** if every pair of elements $a, b \in L$ has both a supremum and an infimum.

- $\sup(a, b)$ is denoted $a \vee b$ (read “ a join b ”).
- $\inf(a, b)$ is denoted $a \wedge b$ (read “ a meet b ”).

Example 14 (Example 10: Lattices vs Non-Lattices). • $(\mathbb{N}, |)$ is a lattice ($\vee = \text{lcm}$, $\wedge = \text{gcd}$).

- $(\mathcal{P}(S), \subseteq)$ is a lattice ($\vee = \cup$, $\wedge = \cap$).
- A chain is always a lattice ($\vee = \max$, $\wedge = \min$).
- **Non-Lattice Example:** The poset in Example 7 where upper bounds e, f are noncomparable. $\sup(b, d)$ does not exist $\implies$ Not a lattice.

5.1 Lattices as Algebraic Structures

Definition 16: Algebraic Definition of Lattice

A lattice is a set L with two binary operations $\vee$ (join) and $\wedge$ (meet) satisfying for all $a, b, c \in L$:

1. **Idempotent:** $a \vee a = a$; $a \wedge a = a$
2. **Commutative:** $a \vee b = b \vee a$; $a \wedge b = b \wedge a$
3. **Associative:** $(a \vee b) \vee c = a \vee (b \vee c)$; $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4. **Absorption:** $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$

Duality Principle

Any theorem derived from the axioms of a lattice remains true if $\vee$ and $\wedge$ are interchanged. This follows from the symmetry of the axioms.

5.2 Sublattices, Products, and Homomorphisms

Definition 17: Sublattice

A subset M of a lattice L is a **sublattice** if M is closed under the operations $\vee$ and $\wedge$ of L . That is, $a, b \in M \implies a \vee b \in M$ and $a \wedge b \in M$.

Definition 18: Product of Lattices

If L_1 and L_2 are lattices, the product $L_1 \times L_2$ is a lattice with operations defined component-wise:

$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \vee b_2)$$

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \wedge b_2)$$

Definition 19: Lattice Homomorphism

A function $f : L \rightarrow L'$ between lattices is a **homomorphism** if it preserves the operations:

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b)$$

A bijective homomorphism is an **isomorphism**.

6 Model Examination Questions

Problem 15 (Problem 1). Define partial order. Show that the relation “divides” ($|$) is not a partial order on the set of integers $\mathbb{Z}$, but is a partial order on $\mathbb{N}$.

Problem 16 (Problem 2). Let $S = \{1, 2, 3, 4, 6, 8, 12, 24\}$. Draw the Hasse diagram of S ordered by divisibility. Identify all minimal, maximal, first, and last elements.

Problem 17 (Problem 3). Consider the poset $P = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ ordered by set inclusion.

1. Draw the Hasse diagram.
2. Find $\sup(\{a\}, \{b\})$ and $\inf(\{a, b\}, \{a, c\})$.
3. Is P a lattice? Justify.

Problem 18 (Problem 4). Define consistent enumeration. Find two different consistent enumerations for the poset defined by the Hasse diagram where a and b are minimal, c covers a and b , and d covers c .

Problem 19 (Problem 5). Let $L = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the set of divisors of 30.

1. Verify that $(L, |)$ is a lattice.
2. Find $10 \vee 15$ and $6 \wedge 15$.
3. Identify the operations $\vee$ and $\wedge$ in terms of number theory functions.

Problem 20 (Problem 6). Prove that in any lattice L , the absorption laws hold: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Problem 21 (Problem 7). Give an example of a poset that is **not** a lattice. Explain why it fails the definition.

Problem 22 (Problem 8). Define lexicographical order. Order the words “bat”, “bad”, “bed”, “apple”, “ape” using the standard alphabet order.

Problem 23 (Problem 9). If $f : L \rightarrow L'$ is a lattice homomorphism, prove that f is order-preserving (i.e., if $a \preceq b$ in L , then $f(a) \preceq f(b)$ in L').

Problem 24 (Problem 10). Explain the difference between a maximal element and a last element in a poset. Give an example of a poset with a maximal element that is not a last element.

All the Best for Your Examination!

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